BIP Relativistic Fluid Dynamics
Kinetic Theory - Exercises

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Exercise 0: Invariance of 4-volume

Prove that $d^3\mathbf{x}\, d^3\mathbf{k}$ is a scalar invariant.
We know that

$$\frac{d^3\mathbf{k}'}{k_0'} = \frac{d^3\mathbf{k}}{k_0}$$

is a scalar invariant, where the primed momenta $\mathbf{k}'$ refer to a different reference frame. Indeed, this is equivalent to the 4-volume with the constraint for the particles to be on shell:

$$d^4k\, \theta(k^0)\delta(k^2 - m^2) = \frac{1}{2}\frac{d^3k}{k^0}$$

We choose the primed frame of reference at rest, where:

$$\mathbf{k}' = 0 \tag{1}$$

and therefore $d^3\mathbf{x}'$ is the proper volume. We know that:

$$d^3\mathbf{x} = \sqrt{1 - v^2}\, d^3\mathbf{x}' = \frac{1}{\gamma}\, d^3\mathbf{x}' \tag{2}$$

On the other hand, from Lorentz transformation:

$$k_0' = \frac{1}{\gamma} k_0 \tag{3}$$

being also $k_0 = k^0$.

If we calculate:

$$d^3\mathbf{x}\, d^3\mathbf{k} = \frac{1}{\gamma} d^3\mathbf{x}' \frac{k_0}{k_0'} d^3\mathbf{k}' = d^3\mathbf{x}'\, d^3\mathbf{k}', \tag{4}$$

by exploiting

$$\frac{1}{\gamma}\frac{k_0}{k_0'} = 1. \tag{5}$$

Thus $d^3\mathbf{x}\, d^3\mathbf{k}$ is a scalar invariant.

Exercise 1: Derivation of the entropy rate

Ref: Kinetic Theory - Lecture 1 (Prof. Plumari).

Recall the definition of entropy 4-flow and entropy rate which have been given in the lectures:

$$S^\mu = -\int \frac{d^3k}{k^0} k^\mu f(x,k)[\log f(x,k) - 1] = -\int dK k^\mu f_k [\log f_k - 1]; \tag{6}$$

$$\zeta = \partial_\mu S^\mu = -\int dK \log f_k\, k^\mu\, \partial_\mu f_k \tag{7}$$

In the RHS we used the compact symbols:

$$f_k \equiv f(x,k), \qquad dK = \frac{d^3k}{k^0}, \tag{8}$$

which will be useful in the following to simplify the notation.

We have the complete expression for the binary collision integral:

$$k^\mu \partial_\mu f_k = \frac{1}{2}\int dK'\, dP\, dP' (f_p f_{p'} W_{pp'|kk'} - f_k f_{k'} W_{kk'|pp'}), \tag{9}$$

thus the expression for the entropy production becomes:

$$\zeta = -\frac{1}{2}\int dK\, dK'\, dP\, dP' \log f_k (f_p f_{p'} W_{pp'|kk'} - f_k f_{k'} W_{kk'|pp'}) = \tag{10}$$

Since we are integrating over momenta, we can split the integral and rename the variables:

$$= -\frac{1}{2}\left[\int dK\, dK'\, dP\, dP' \log f_k (f_p f_{p'} W_{pp'|kk'}) - \int dK\, dK'\, dP\, dP' \log f_k (f_k f_{k'} W_{kk'|pp'})\right] = \tag{11}$$

$$= -\frac{1}{2}\left[\int dK\, dK'\, dP\, dP' \log f_k (f_p f_{p'} W_{pp'|kk'}) - \int dP\, dP'\, dK\, dK' \log f_p (f_p f_{p'} W_{pp'|kk'})\right] = \tag{12}$$

$$= -\frac{1}{2}\int dK\, dK'\, dP\, dP' (\log f_k - \log f_p) f_p f_{p'} W_{pp'|kk'} \tag{13}$$

We can go even further in the symmetrization, since one can exchange $k \longleftrightarrow k'$ and $p \longleftrightarrow p'$ in the transition rate: $W_{pp'|kk'} = W_{p'p|k'k}$. One immediately gets:

$$= -\frac{1}{4}\int dK\, dK'\, dP\, dP' [\log(f_k f_{k'}) - \log(f_p f_{p'})] f_p f_{p'} W_{pp'|kk'} \tag{14}$$

We can still exploit microreversibility:

$$W_{pp'|kk'} = W_{kk'|pp'}.$$

If we multiply by $f_k f_{k'}$ both sides and integrate over momenta:

$$\int dK\, dK'\, dP\, dP' f_k\, f_{k'} W_{pp'|kk'} = \int dK\, dK'\, dP\, dP' f_k\, f_{k'} W_{kk'|pp'} \tag{15}$$

$$\int dK\, dK'\, dP\, dP' f_k\, f_{k'} W_{pp'|kk'} = \int dK\, dK'\, dP\, dP' f_p\, f_{p'} W_{pp'|kk'} \tag{16}$$

$$\int dK\, dK'\, dP\, dP' (f_k\, f_{k'} - f_p f_{p'}) W_{pp'|kk'} = 0 \tag{17}$$

Now we can just sum the latter equation and Eq. (14), to get the most symmetrized version of the entropy rate:

$$\zeta = -\frac{1}{4}\int dK\, dK'\, dP\, dP' \left\{[\log(f_k f_{k'}) - \log(f_p f_{p'})]\, f_p f_{p'} + f_p f_{p'} - f_k f_{k'}\right\} W_{pp'|kk'} = \tag{18}$$

$$= -\frac{1}{4}\int dK\, dK'\, dP\, dP' \left[\log\left(\frac{f_k f_{k'}}{f_p f_{p'}}\right) - \frac{f_k f_{k'}}{f_p f_{p'}} + 1\right] f_p f_{p'} W_{pp'|kk'}, \tag{19}$$

$$\tag{20}$$

This is the expression we were looking for, since now the integrand is manifestly non-negative, due to the fact that also the distribution function and the transition rate W are positive.

Exercise 2: Determination of a and b (Jüttner Distribution)

Ref: Kinetic Theory - Lecture 1 (Prof. Plumari).

Prove that $b^\mu \propto u^\mu$

We know that the particle 4-flow:

$$N^\mu = \int dK\, k^\mu f_k = e^a \int dK\, e^{b_\mu k^\mu} \tag{21}$$

This integral can converge only if $b_\mu k^\mu < 0$. We define $B_\mu = -b_\mu$ and the function:

$$Z(B) = \int dK\, e^{-B_\mu k^\mu} \tag{22}$$

This quantity is by construction a Lorentz scalar and furthermore the following relationships hold:

$$\frac{\partial Z(B)}{\partial B^\mu} = -\int dK\, k^\mu\, e^{-B_\nu k^\nu} \implies N^\mu = -e^a \frac{\partial Z(B)}{\partial B^\mu} \tag{23}$$

However, since $Z(B)$ is a scalar quantity, it must depend only on $B^2 = B_\mu B^\mu$, therefore:

$$\frac{\partial Z}{\partial B^\mu} = \frac{\partial B^2}{\partial B^\mu}\frac{\partial Z}{\partial B^2} = 2B^\mu \frac{\partial Z}{\partial B^2} \propto N^\mu = n u^\mu \tag{24}$$

Therefore, we can say $B^\mu = \beta\, u^\mu$ and $f_k \propto \exp(\alpha - \beta\, u_\mu k^\mu)$

Find a and β

Let's collect some results:

$$n = N^\mu u_\mu = e^a \int dK\, (k^\mu u_\mu) \exp(-\beta\, k^\nu u_\nu), \tag{25}$$

$$\varepsilon = T^{\mu\nu} u_\mu u_\nu = \int dK\, (k^\mu u_\mu)^2 \exp(-\beta\, k^\nu u_\nu) \tag{26}$$

$$p = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} = -\frac{1}{3}\int dK\, \Delta_{\mu\nu}\, k^\mu k^\nu f_k = -\frac{e^a}{3}\int dK\, (k^2 - m^2) \exp(-\beta\, k^\nu u_\nu) \tag{27}$$

In the last step we have exploited:

$$k^\mu k^\nu \Delta_{\mu\nu} = k^\mu k^\nu (g_{\mu\nu} - u_\mu u_\nu) = k^2 - (k^\mu u_\mu)^2 = k^2 - k_0^2 = -|\mathbf{k}|^2, \tag{28}$$

where we have assumed to be in LRF where $u^\mu = (1, \mathbf{0})$.

Now let us consider the entropy density:

$$s = S^\mu u_\mu = -\int dK\, k^\mu u_\mu\, f\, (\log f - 1) = -\int dK\, k^\mu u_\mu\, f (a - \beta\, k^\mu u_\mu - 1) = -an + \beta\, \varepsilon + n \tag{29}$$

From thermodynamics we know:

$$\mu\, n = \varepsilon - Ts + p \implies \mu = \frac{\varepsilon}{n} - \frac{Ts}{n} + T \tag{30}$$

If one compares this expression with Eq. (29), one simply identifies:

$$a = \frac{\mu}{T}, \qquad \beta = \frac{1}{T} \tag{31}$$

Explicit expression of thermodynamic quantities

Reminder. Second-order modified Bessel functions:

$$K_n(z) = \frac{2^n n!}{(2n)!}\frac{1}{z^n}\int_z^\infty d\tau\, (\tau^2 - z^2)^{n-\frac{1}{2}} e^{-\tau} = \frac{2^{n-1}(n-1)!}{(2n-2)!}\frac{1}{z^n}\int_z^\infty d\tau\, (\tau^2 - z^2)^{n-\frac{3}{2}}\, \tau\, e^{-\tau}. \tag{32}$$

Particle density

According to the definition

$$n = N^{\mu} u_{\mu} = e^{a} \int dK \, (k^{\mu} u_{\mu}) \, \exp(-\beta \, k^{\mu} u_{\mu})$$

Let us calculate this quantity in the LRF, where $u^{\mu} = (1, \mathbf{0})$. In order to reduce the integral in the same shape of Eq. (32), let us also perform the change of variable:

$$z = \beta \, m, \qquad \tau = \beta \, k^{0}, \tag{33}$$

$$k_0^2 = k^2 + m^2 \implies k^2 = \frac{\tau^2 - z^2}{\beta^2}, \qquad dk = \frac{\tau}{\beta \sqrt{\tau^2 - z^2}} d\tau. \tag{34}$$

Therefore:

$$n = e^{a} \int \frac{d^3\mathbf{k}}{k^0} k^0 e^{-\beta k^0} = e^{a} \int d\Omega \int dk \, k^2 e^{-\beta k^0} \tag{35}$$

Using Eq. (32) for the Bessel functions, the expression for the particle density can be recast as:

$$n = 4\pi \, e^{a} \frac{m^2}{\beta} K_2(z) = \frac{4\pi e^{a} z^2}{\beta^3} K_2(z) \tag{36}$$

Pressure

Starting from the expression previously found for the pressure and performing the same change of variables, one finds:

$$p = \frac{1}{3} \int \frac{d^3\mathbf{k}}{k^0} k^2 \, f_k = \frac{1}{3} e^{a} \int d\Omega \int \frac{dk}{k^0} k^4 e^{-\beta k^0} = \tag{37}$$

$$= \frac{4\pi \, e^{a}}{3} \int_z^{\infty} \frac{d\tau}{\tau/\beta} \frac{\tau}{\beta\sqrt{\tau^2 - z^2}} \frac{(\tau^2 - z^2)^2}{\beta^4} e^{-\tau} = \frac{4\pi e^{a}}{\beta^4} z^2 K_2(z) = \frac{n}{\beta} \tag{38}$$

Energy density

According to the same strategy:

$$\varepsilon = e^{a} \int \frac{d^3\mathbf{k}}{k^0} (\mathbf{k}^2 + m^2) e^{-\beta k^0} = 4\pi \, e^{a} \left(\int \frac{dk}{k^0} k^2 e^{-\beta k^0} + \int \frac{dk}{k^0} m^2 e^{-\beta k^0} \right) = \tag{39}$$

$$= 4\pi \, e^{a} \left(\int \frac{d\tau}{\tau/\beta} \frac{\tau}{\beta\sqrt{\tau^2 - z^2}} \frac{(\tau^2 - z^2)^2}{\beta^4} e^{-\tau} + m^2 \int \frac{d\tau}{\tau/\beta} \frac{\tau}{\beta\sqrt{\tau^2 - z^2}} \frac{\tau^2 - z^2}{\beta^2} e^{-\tau} \right) = \tag{40}$$

$$= 4\pi \, e^{a} \left[\frac{3z^2}{\beta^4} K_2(z) + \frac{z}{\beta^2} K_1(z) \right] = 4\pi \, e^{a} \, m^4 \left[\frac{3K_2(z)}{z^2} + \frac{K_1(z)}{z} \right] \tag{41}$$

Exercise 3: Boltzmann Equation at 1st order in CE

Ref: Kinetic Theory - Lecture 2 (Chapman–Enskog) (Prof. Plumari).

Exercise 3.0

Prove that, at equilibrium,

$$f_{0k} f_{0k'} \tilde{f}_{0p} \tilde{f}_{0p'} = f_{0p} f_{0p'} \tilde{f}_{0k} \tilde{f}_{0k'}, \tag{42}$$

given the definitions in the semiclassical case:

$$f_{0k} = \frac{1}{e^{\beta_{\mu} k^{\mu} - \alpha} + a}, \qquad \tilde{f}_k = 1 - a f_k \tag{43}$$

It is trivial using the equilibrium distribution and the conservation of 4-momentum:

$$\frac{1}{e^{\beta_\mu k^\mu - \alpha} + a} \frac{1}{e^{\beta_\mu k'^\mu - \alpha} + a} e^{\beta_\mu p^\mu - \alpha} e^{\beta_\mu p'^\mu - \alpha} = \frac{1}{e^{\beta_\mu p^\mu - \alpha} + a} \frac{1}{e^{\beta_\mu p'^\mu - \alpha} + a} e^{\beta_\mu k^\mu - \alpha} e^{\beta_\mu k'^\mu - \alpha}. \tag{44}$$

0.1 Exercise 3.1

Derive the first-order term of collision integral $C^{(1)}$ in Chapman-Enskog (Ref. Chapman-Enskog slides of Prof. Plumari).

Let us start with the Boltzmann Equation written in a covariant way:

$$Df_k = \frac{1}{E_k} k^\mu \nabla_\mu f_k = \frac{1}{E_k} C[f_k] \tag{45}$$

If we define the dimensionless derivatives:

$$\lambda D = \mathrm{Kn}\, \hat{D}, \qquad \lambda \nabla_\mu = \mathrm{Kn}\, \hat{\nabla}_\mu \tag{46}$$

If we multiply Eq. (45) by the mean free path λ we get:

$$\mathrm{Kn}\, \hat{D} f_k = \frac{\mathrm{Kn}}{E_k} k^\mu \hat{\nabla}_\mu f_k = \frac{\lambda}{E_k} C[f_k] \tag{47}$$

Now we expand the distribution function f_k around equilibrium $f_k^{(0)}$:

$$f_k = f_k^{(0)} + \mathrm{Kn}\, f_k^{(1)} + \mathrm{Kn}^2\, f_k^{(2)} + \dots \tag{48}$$

Consider now the full expression for the binary collision integral:

$$C[f_k] = \frac{1}{2} \int dK'\, dP\, dP' W_{kk'|pp'} (f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} - f_k f_{k'} \tilde{f}_p \tilde{f}_{p'}) \tag{49}$$

In principle one can think to expand this quantity in powers of Kn:

$$C[f_k] = C^{(0)} + \mathrm{Kn}\, C^{(1)} + \mathrm{Kn}^2\, C^{(2)} + \dots \tag{50}$$

Trivially, the zero-order term corresponds to the collision integral in which the distribution function is at equilibrium.
Let us now compute the first order. In both addends inside the parenthesis we can include one power of Kn at once:

$$C^{(1)} = \frac{1}{2} \int dK'\, dP\, dP' W_{kk'|pp'} (f_p^{(1)} f_{p'}^{(0)} \tilde{f}_k^{(0)} \tilde{f}_{k'}^{(0)} + \dots - f_k^{(0)} f_{k'}^{(1)} \tilde{f}_p^{(0)} \tilde{f}_{p'}^{(0)}) \tag{51}$$

Collecting everything and using $\tilde{f}_k = 1 - a f_k$:

$$\begin{aligned} C^{(1)} = \frac{1}{2} \int dK'\, dP\, dP' W_{kk'|pp'} \Bigg[& f_p^{(0)} f_{p'}^{(0)} \tilde{f}_k^{(0)} \tilde{f}_{k'}^{(0)} \left(\frac{f_p^{(1)}}{f_p^{(0)}} + \frac{f_{p'}^{(1)}}{f_{p'}^{(0)}} - a \frac{\tilde{f}_k^{(1)}}{\tilde{f}_k^{(0)}} - a \frac{\tilde{f}_{k'}^{(1)}}{\tilde{f}_{k'}^{(0)}} \right) + \\ & - f_k^{(0)} f_{k'}^{(0)} \tilde{f}_p^{(0)} \tilde{f}_{p'}^{(0)} \left(\frac{f_k^{(1)}}{f_k^{(0)}} + \frac{f_{k'}^{(1)}}{f_{k'}^{(0)}} - a \frac{\tilde{f}_p^{(1)}}{\tilde{f}_p^{(0)}} - a \frac{\tilde{f}_{p'}^{(1)}}{\tilde{f}_{p'}^{(0)}} \right) \Bigg] \end{aligned} \tag{52}$$

We can use the result previously found in Eq. (42):

$$C^{(1)} = -\frac{1}{2} \int dK'\, dP\, dP' W_{kk'|pp'} f_k^{(0)} f_{k'}^{(0)} \tilde{f}_p^{(0)} \tilde{f}_{p'}^{(0)} \left(\frac{f_p^{(1)}}{f_p^{(0)}} + \frac{f_{p'}^{(1)}}{f_{p'}^{(0)}} - a \frac{\tilde{f}_k^{(1)}}{\tilde{f}_k^{(0)}} - a \frac{\tilde{f}_{k'}^{(1)}}{\tilde{f}_{k'}^{(0)}} - \frac{f_k^{(1)}}{f_k^{(0)}} - \frac{f_{k'}^{(1)}}{f_{k'}^{(0)}} + a \frac{\tilde{f}_p^{(1)}}{\tilde{f}_p^{(0)}} + a \frac{\tilde{f}_{p'}^{(1)}}{\tilde{f}_{p'}^{(0)}} \right) \tag{53}$$

Now we simply exploit:

$$\frac{f_k^{(1)}}{f_k^{(0)}} - a \frac{\tilde{f}_k^{(1)}}{\tilde{f}_k^{(0)}} = \frac{(1 - a f_k^{(0)}) f_k^{(1)} + a \tilde{f}_k^{(1)} f_k^{(0)}}{f_k^{(0)} \tilde{f}_k^{(0)}} = \frac{f_k^{(1)}}{f_k^{(0)} \tilde{f}_k^{(0)}}. \tag{54}$$

And finally we find:

$$C[f_k]^{(1)} = -\frac{1}{2} \int dK'\, dP\, dP' W_{kk'|pp'} f_k^{(0)} f_{k'}^{(0)} \tilde{f}_p^{(0)} \tilde{f}_{p'}^{(0)} \left(\frac{f_k^{(1)}}{f_k^{(0)} \tilde{f}_k^{(0)}} + \frac{f_{k'}^{(1)}}{f_{k'}^{(0)} \tilde{f}_{k'}^{(0)}} - \frac{f_p^{(1)}}{f_p^{(0)} \tilde{f}_p^{(0)}} - \frac{f_{p'}^{(1)}}{f_{p'}^{(0)} \tilde{f}_{p'}^{(0)}} \right) \tag{55}$$

Exercise 4: First order CE equations for primary hydrodynamic variables

Ref: Kinetic Theory - Lecture 2 (Chapman–Enskog) (Prof. Plumari).

Starting from the definitions:

$$I_{nq} \equiv \frac{(-1)^q}{(2q+1)!!}\int dK f_{0\mathbf{k}} k_0^{n-2q}(m^2-k_0^2)^q \tag{56}$$

$$\equiv \frac{(-1)^q}{(2q+1)!!}\left\langle (k^\mu u_\mu)^{n-2q}(\Delta^{\mu\nu}k_\mu k_\nu)^q\right\rangle_0 , \tag{57}$$

$$J_{nq} \equiv \frac{(-1)^q}{(2q+1)!!}\int dK f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} k_0^{n-2q}(m^2-k_0^2)^q \tag{58}$$

$$\equiv \frac{(-1)^q}{(2q+1)!!}\left\langle \tilde{f}_{0\mathbf{k}}(k^\mu u_\mu)^{n-2q}(\Delta^{\mu\nu}k_\mu k_\nu)^q\right\rangle_0 , \tag{59}$$

$$D_{ij} = J_{n+1,q}\,J_{n-1,q} - J_{nq}^2 \tag{60}$$

and the hydrodynamic equations:

$$Dn + n\theta + \nabla_\mu n^\mu - n^\mu Du_\mu = 0, \tag{61}$$

$$D\varepsilon + (\varepsilon + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{62}$$

$$(\varepsilon + P_0 + \Pi)Du^\alpha - \nabla^\alpha(P_0+\Pi) - \pi^{\alpha\beta}Du_\beta + \Delta^\alpha{}_\nu\nabla_\mu\pi^{\mu\nu} = 0, \tag{63}$$

prove that, at first order in Kn:

$$\hat{D}\alpha_0 = \frac{(\varepsilon_0+P_0)J_{20} - n_0J_{30}}{D_{20}}\hat{\theta} + \mathcal{O}(\mathrm{Kn}^2), \tag{64}$$

$$\hat{D}\beta_0 = \frac{(\varepsilon_0+P_0)J_{10} - n_0J_{20}}{D_{20}}\hat{\theta} + \mathcal{O}(\mathrm{Kn}^2), \tag{65}$$

$$\hat{D}u^\mu = \frac{1}{\varepsilon_0+P_0}\hat{\nabla}^\mu P_0 + \mathcal{O}(\mathrm{Kn}^2). \tag{66}$$

Starting from the definition of the distribution function $f_{0k}(\alpha_0,\beta_0) = \exp(\alpha_0 - \beta_0 k^0)$
If one differentiates:

$$dI_{nq} = \frac{\partial I_{nq}}{\partial\alpha_0}d\alpha_0 + \frac{\partial I_{nq}}{\partial\beta_0}d\beta_0 \tag{67}$$

α_0 and β_0 appear only inside the f_{0k}. Indeed:

$$\frac{\partial f_{0k}}{\partial\alpha_0} = \frac{\partial}{\partial\alpha_0}\frac{1}{\exp(\beta_0k^0-\alpha_0)+a} = \frac{\exp(\beta_0k^0-\alpha_0)}{[\exp(\beta_0k^0-\alpha_0)+a]^2} = f_{0k}\tilde{f}_{0k} \tag{68}$$

$$\frac{\partial f_{0k}}{\partial\beta_0} = \frac{\partial}{\partial\beta_0}\frac{1}{\exp(\beta_0k^0-\alpha_0)+a} = -k^0\frac{\exp(\beta_0k^0-\alpha_0)}{[\exp(\beta_0k^0-\alpha_0)+a]^2} = -k^0 f_{0k}\tilde{f}_{0k} \tag{69}$$

Therefore:

$$dI_{nq} = J_{nq}d\alpha_0 - J_{n+1,q}d\beta_0. \tag{70}$$

The fluid dynamic quantities ε, n and P_0 can be expressed in terms of the previously defined integrals. Indeed:

$$n = I_{10}, \quad \varepsilon = I_{20}, \quad P_0 = I_{21} \tag{71}$$

We have to invert the set of equations (70) to find $d\alpha_0$ and $d\beta_0$. One can take for instance the case corresponding to the particle and energy density:

$$dn = J_{10}\,d\alpha_0 - J_{20}\,d\beta_0, \tag{72}$$

$$d\varepsilon = J_{20}\,d\alpha_0 - J_{30}\,d\beta_0 \tag{73}$$

This leads to:

$$d\beta_0 = -\frac{J_{10}\, d\varepsilon}{D_{20}} + \frac{J_{20}\, dn}{D_{20}}, \tag{74}$$

$$d\alpha_0 = \frac{J_{30}\, dn}{D_{20}} - \frac{J_{20}\, d\varepsilon}{D_{20}} \tag{75}$$

Now we can just use it along with the hydrodynamic equations:

$$\hat{D}\alpha_0 = \frac{1}{D_{20}} \left\{ \left[(\varepsilon_0 + P_0)J_{20} - n_0 J_{30}\right] \hat{\theta} + J_{20} \left(\Pi\hat{\theta} - \pi^{\alpha\beta}\hat{\sigma}_{\alpha\beta} \right) - J_{30} \left(\hat{\nabla}_\mu n^\mu - n^\mu \hat{D} u_\mu \right) \right\}, \tag{76}$$

$$\hat{D}\beta_0 = \frac{1}{D_{20}} \left\{ \left[(\varepsilon_0 + P_0)J_{10} - n_0 J_{20}\right] \hat{\theta} + J_{10} \left(\Pi\hat{\theta} - \pi^{\alpha\beta}\hat{\sigma}_{\alpha\beta} \right) - J_{20} \left(\hat{\nabla}_\mu n^\mu - n^\mu \hat{D} u_\mu \right) \right\} \tag{77}$$

Remembering that all the dissipative quantities are already proportional to gradients, one can stop at the first order in the gradients themselves, finding:

$$\hat{D}\alpha_0 = \frac{1}{D_{20}} \left\{ \left[(\varepsilon_0 + P_0)J_{20} - n_0 J_{30}\right] \hat{\theta} \right\}, \tag{78}$$

$$\hat{D}\beta_0 = \frac{1}{D_{20}} \left\{ \left[(\varepsilon_0 + P_0)J_{10} - n_0 J_{20}\right] \hat{\theta} \right\}, \tag{79}$$

$$\hat{D}u^\alpha = \frac{\nabla^\alpha P_0}{\varepsilon + P_0 + \Pi} \tag{80}$$

Exercise 5: Viscous corrections to second order in moments

Ref: Kinetic Theory - Lecture 2 (Israel–Stewart) (Prof. Plumari).

Consider the expansion in momentum moments of the particle current and the energy-momentum tensor:

$$N^\mu = N_0^\mu + \epsilon J_0^\mu + J_0^{\mu\nu}\epsilon_\nu + J_0^{\mu\nu\lambda}\epsilon_{\nu\lambda} + \dots \tag{81}$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \epsilon J_0^{\mu\nu} + J_0^{\mu\nu\lambda}\epsilon_\lambda + J_0^{\mu\nu\lambda\rho}\epsilon_{\lambda\rho} + \dots \tag{82}$$

where

$$\begin{aligned} I_0^{\alpha_1 \dots \alpha_n} &\equiv \int dK\, k^{\alpha_1} \dots k^{\alpha_n} f_{0\mathbf{k}}\,, \\ J_0^{\alpha_1 \dots \alpha_n} &\equiv \int dK\, k^{\alpha_1} \dots k^{\alpha_n} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}\,. \end{aligned}$$

The following equations hold:

$$\begin{aligned} I_0^\mu &= I_{10} u^\mu\,, \\ I_0^{\mu\nu} &= I_{20} u^\mu u^\nu - I_{21} \Delta^{\mu\nu}\,, \\ J_0^\mu &= J_{10} u^\mu\,, \\ J_0^{\mu\nu} &= J_{20} u^\mu u^\nu - J_{21} \Delta^{\mu\nu}\,, \\ J_0^{\mu\nu\lambda} &= J_{30} u^\mu u^\nu u^\lambda - 3 J_{31} u^{(\mu} \Delta^{\nu\lambda)}\,, \\ J_0^{\mu\nu\lambda\rho} &= J_{40} u^\mu u^\nu u^\lambda u^\rho - 6 J_{41} u^{(\mu} u^\nu \Delta^{\lambda\rho)} + 3 J_{42} \Delta^{(\mu\nu} \Delta^{\lambda\rho)} \end{aligned} \tag{83}$$

Prove:

$$\begin{aligned} n^\mu &= -J_{21} \Delta^{\mu\nu} \epsilon_\nu - 2 J_{31} \Delta^{\mu\nu} u^\lambda \epsilon_{\nu\lambda}\,, \\ \pi^{\mu\nu} &= 2 J_{42} \Delta^{\mu\nu}_{\lambda\rho} \epsilon^{\lambda\rho}\,, \\ \Pi &= J_{21} \epsilon + J_{31} u^\lambda \epsilon_\lambda + \left(J_{41} + \frac{5}{3} J_{42} \right) u^\lambda u^\rho \epsilon_{\lambda\rho}\,, \end{aligned}$$

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Diffusion current

We have decomposed

$$N^\mu = n\,u^\mu + n^\mu \implies n_\mu = \Delta_{\mu\nu} N^\nu \tag{84}$$

Let us use the expansion above, stopping at quadratic order in the moments:

$$n_\mu = \Delta_{\mu\nu} N_0^\nu + \epsilon\,\Delta_{\mu\nu}\,J_0^\nu + \Delta_{\mu\nu}\,J_0^{\nu\rho}\,\epsilon_\rho + \Delta_{\mu\nu} J_0^{\nu\lambda\rho}\,\epsilon_{\lambda\rho} \tag{85}$$

We exploit the properties of the projector:

$$\Delta_{\mu\nu} J_0^\mu = (g_{\mu\nu} - u_\mu u_\nu) J_{10} u^\mu = J_{10}(u_\nu - u_\nu) = 0 \tag{86}$$

In general $\Delta^{\mu\nu} u_\mu = 0$, and we will use this relation in the following.

For the second term:

$$\Delta_{\mu\nu} J^{\nu\rho} = \Delta_{\mu\nu}(J_{20} u^\nu\, u^\rho - J_{21}\Delta^{\nu\rho}) = -J_{21}\Delta_{\mu\nu}\Delta^{\nu\rho} = -J_{21}\Delta^\rho_\mu \tag{87}$$

Finally for the third term:

$$\Delta_{\mu\nu} J_0^{\nu\rho\lambda} = \Delta_{\mu\nu}(J_{30}\,u^\nu u^\lambda u^\rho - 3J_{31} u^{(\nu}\Delta^{\lambda\rho)}) = -3J_{31}\Delta_{\mu\nu}\frac{1}{3!}(2u^\nu\Delta^{\lambda\rho} + 2u^\rho\Delta^{\nu\lambda} + 2u^\lambda\Delta^{\rho\nu}) \tag{88}$$

since we are going to multiply this quantity times the symmetric tensor $\epsilon_{\lambda\rho}$ and exploiting the orthogonality of u^μ and $\Delta^{\mu\nu}$:

$$\Delta_{\mu\nu} J_0^{\nu\lambda\rho}\epsilon_{\lambda\rho} = -2J_{31}\Delta^{\mu\nu}u^\lambda\epsilon_{\nu\lambda} \tag{89}$$

Shear stress tensor

We know that

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} = \Delta^{\mu\nu}_{\alpha\beta}(I_0^{\alpha\beta} + \epsilon J_0^{\alpha\beta} + J_0^{\alpha\beta\gamma}\epsilon_\gamma + J_0^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}) \tag{90}$$

Remind that:

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta} \tag{91}$$

and the contraction with the rank-2 projector gives:

$$\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta} = \left[\frac{1}{2}(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right]\Delta^{\alpha\beta} = \Delta^{\mu\nu} - \Delta^{\mu\nu} = 0 \tag{92}$$

where we have used $\Delta^{\alpha\beta}\Delta_{\beta\gamma} = \Delta^\alpha_\gamma$ and $\Delta^{\alpha\beta}\Delta_{\alpha\beta} = 3$.

This means that the first two terms of Eq. (90) are vanishing since they are proportional to a sum of the 4-velocity u^μ and of the projector $\Delta^{\mu\nu}$.

Let us consider now the third term:

$$\left[\frac{1}{2}(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right]\frac{2}{3!}(u^\alpha\Delta^{\beta\gamma} + u^\gamma\Delta^{\alpha\beta} + u^\beta\Delta^{\gamma\alpha}) = u^\gamma\Delta^{\mu\nu} - u^\gamma\Delta^{\mu\nu} = 0 \tag{93}$$

Let us look now at the fourth term:

$$\Delta^{\mu\nu}_{\alpha\beta}\Delta^{(\alpha\beta}\Delta^{\gamma\delta)} = \Delta^{\mu\nu}_{\alpha\beta}\frac{1}{4!}(8\Delta^{\alpha\beta}\Delta^{\gamma\delta} + 8\Delta^{\alpha\gamma}\Delta^{\beta\delta} + 8\Delta^{\alpha\delta}\Delta^{\gamma\beta}) = \frac{2}{3}\Delta^{\mu\nu\gamma\delta} \tag{94}$$

And eventually:

$$\pi^{\mu\nu} = 2J_{42}\Delta^{\mu\nu\gamma\delta}\epsilon_{\gamma\delta} \tag{95}$$

Bulk viscous pressure

We know that Π is difference between the spatial trace of the energy momentum tensor and its value at equilibrium P_0, which means:

$$\Pi = -\frac{1}{3}\Delta_{\alpha\beta}(T^{\alpha\beta} - T_0^{\alpha\beta}) = -\frac{1}{3}\Delta_{\alpha\beta}(\epsilon J_0^{\alpha\beta} + J_0^{\alpha\beta\gamma}\epsilon_\gamma + J_0^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}) = \epsilon\, J_{21} + J_{31}u^\gamma\epsilon_\gamma + \left(J_{41} + \frac{5}{3}J_{42}\right)u^\gamma u^\delta\epsilon_{\gamma\delta}. \tag{96}$$

Step by step, using the orthogonality of u_μ and $\Delta^{\mu\nu}$:

- $$\Delta_{\alpha\beta}J_0^{\alpha\beta} = \Delta_{\alpha\beta}(J_{20}u^\alpha u^\beta - J_{21}\Delta^{\alpha\beta}) = -3J_{21} \tag{97}$$

- $$\Delta_{\alpha\beta}J_0^{\alpha\beta\gamma} = \Delta_{\alpha\beta}\left[J_{30}u^\alpha u^\beta u^\gamma - 3J_{31}\frac{1}{3!}(2u^\alpha\Delta^{\beta\gamma} + 2u^\beta\Delta^{\gamma\alpha} + 2u^\gamma\Delta^{\alpha\beta})\right] = -3J_{31}u^\gamma \tag{98}$$

- $$\begin{aligned}\Delta_{\alpha\beta}J_0^{\alpha\beta\gamma\delta} &= \Delta_{\alpha\beta}\left[J_{40}u^\alpha u^\beta u^\gamma u^\delta - 6J_{41}u^{(\alpha}u^\beta\Delta^{\gamma\delta)} + 3J_{42}\Delta^{(\alpha\beta}\Delta^{\gamma\delta)}\right] = && (99)\\ &= -6J_{41}\Delta_{\alpha\beta}\frac{1}{4!}(4u^\alpha u^\beta\Delta^{\gamma\delta} + 4u^\gamma u^\delta\Delta^{\alpha\beta} + 4u^\alpha u^\gamma\Delta^{\beta\delta} + 4u^\beta u^\delta\Delta^{\alpha\gamma} + 4u^\beta u^\gamma\Delta^{\alpha\delta} + 4u^\alpha u^\delta\Delta^{\beta\gamma}) = && (100)\\ &= -J_{41}\Delta_{\alpha\beta}u^\gamma u^\delta\Delta^{\alpha\beta} = -3J_{41}u^\gamma u^\delta. && (101)\end{aligned}$$

- $$\Delta_{\alpha\beta}\,3J_{42}\frac{1}{4!}(8\Delta^{\alpha\beta}\Delta^{\gamma\delta} + 8\Delta^{\alpha\gamma}\Delta^{\beta\delta} + 8\Delta^{\alpha\delta}\Delta^{\beta\gamma}) = J_{42}(3\Delta^{\gamma\delta} + \Delta^{\gamma\delta} + \Delta^{\gamma\delta}) = 5\Delta^{\gamma\delta} \tag{102}$$

  This quantity has to be multiplied by $-\frac{1}{3}\epsilon_{\gamma\delta}$:

  $$-\frac{5}{3}\Delta^{\gamma\delta}\epsilon_{\gamma\delta} = -\frac{5}{3}(g^{\gamma\delta} - u^\gamma u^\delta)\epsilon_{\gamma\delta} = \frac{5}{3}u^\gamma u^\delta\epsilon_{\gamma\delta}, \tag{103}$$

  exploiting the fact that $\epsilon_{\gamma\delta}$ is traceless.

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