Tutoring 28/05

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EXERCISE 1

In this exercise we show that the operator

$$\widehat{\rho} = \frac{1}{Z} \mathrm{e}^{-\lambda \widehat{A}} \tag{1}$$

maximizes the entropy functional under the constraint that the average value $\langle \widehat{A} \rangle$ is fixed.

The functional we have to maximize is

$$F[\,\widehat{\rho}\,] = -\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho}) - c\ (\mathrm{Tr}\widehat{\rho} - 1) - \lambda \left(\mathrm{Tr}(\widehat{\rho}\,\widehat{A}) - \langle \widehat{A} \rangle \right) \ . \tag{2}$$

Here c and λ are Lagrange multipliers. The first constraint imposes the normalization of the density operator, while the second one fixes the expectation value of $\widehat{A}$:

$$\mathrm{Tr}\widehat{\rho} = 1 \ , \qquad \mathrm{Tr}(\widehat{\rho}\,\widehat{A}) = \langle \widehat{A} \rangle \ .$$

At the maximum, the functional is stationary, namely

$$\frac{\delta F[\,\widehat{\rho}\,]}{\delta \widehat{\rho}} = 0 \ .$$

Let us now introduce the operator

$$\widehat{X} \equiv \widehat{\mathrm{I}} - \widehat{\rho} \ . \tag{3}$$

This rewriting is useful because it allows us to compute the variation of the functional by considering the change

$$\widehat{X} \longrightarrow \widehat{X} + \delta \widehat{X} \ .$$

Since $\widehat{\rho} = \widehat{\mathrm{I}} - \widehat{X}$, this corresponds to

$$\widehat{\rho} \longrightarrow \widehat{\rho} - \delta \widehat{X} \ .$$

In terms of $\widehat{X}$, the functional (2) becomes

$$F[\,\widehat{X}\,] = -\mathrm{Tr}\left[(\widehat{\mathrm{I}} - \widehat{X}) \ln(\widehat{\mathrm{I}} - \widehat{X}) \right] - c \left[\mathrm{Tr}(\widehat{\mathrm{I}} - \widehat{X}) - 1 \right] - \lambda \left[\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X}) \widehat{A} \right) - \langle \widehat{A} \rangle \right] \ . \tag{4}$$

Equivalently, using the linearity and cyclicity of the trace, the last term can also be written as

$$\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X}) \widehat{A} \right) = \mathrm{Tr}\widehat{A} - \mathrm{Tr}(\widehat{X}\widehat{A}) \ . \tag{5}$$

We now compute the finite difference

$$F[\,\widehat{X} + \delta \widehat{X}\,] - F[\,\widehat{X}\,] \tag{6}$$

to first order in $\delta \widehat{X}$. Explicitly,

$$\begin{aligned} F[\,\widehat{X} + \delta \widehat{X}\,] - F[\,\widehat{X}\,] = & -\mathrm{Tr}\left[(\widehat{\mathrm{I}} - \widehat{X} - \delta \widehat{X}) \ln(\widehat{\mathrm{I}} - \widehat{X} - \delta \widehat{X}) \right] \\ & + \mathrm{Tr}\left[(\widehat{\mathrm{I}} - \widehat{X}) \ln(\widehat{\mathrm{I}} - \widehat{X}) \right] \\ & - c \left[\mathrm{Tr}(\widehat{\mathrm{I}} - \widehat{X} - \delta \widehat{X}) - 1 \right] + c \left[\mathrm{Tr}(\widehat{\mathrm{I}} - \widehat{X}) - 1 \right] \\ & - \lambda \left[\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X} - \delta \widehat{X}) \widehat{A} \right) - \langle \widehat{A} \rangle \right] \\ & + \lambda \left[\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X}) \widehat{A} \right) - \langle \widehat{A} \rangle \right] \ . \end{aligned} \tag{7}$$

Let us first consider the entropy term. Up to first order in $\delta\widehat{X}$,

$$
\begin{aligned}
&-\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})\right]+\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X})\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\\
&=\operatorname{Tr}\left[\delta\widehat{X}\,\ln(\widehat{\mathrm{I}}-\widehat{X})\right]-\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}+\mathcal{O}(\delta\widehat{X}^2)\;.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
F[\,\widehat{X}+\delta\widehat{X}\,]-F[\,\widehat{X}\,]=\operatorname{Tr}\left[\delta\widehat{X}\,\ln(\widehat{\mathrm{I}}-\widehat{X})\right]-\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}\\
+c\operatorname{Tr}(\delta\widehat{X})+\lambda\operatorname{Tr}(\delta\widehat{X}\,\widehat{A})+\mathcal{O}(\delta\widehat{X}^2)\;.
\end{aligned}
$$

We now have to compute the variation of the logarithm. The logarithm of an operator can be formally defined through its Taylor expansion. In particular, for sufficiently small $\widehat{X}$ one has

$$
\ln(\widehat{\mathrm{I}}-\widehat{X})=-\sum_{n=1}^{\infty}\frac{\widehat{X}^n}{n}\;. \tag{8}
$$

Hence:

$$
\begin{aligned}
&\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\\
&=-\sum_{n=1}^{\infty}\frac{(\widehat{X}+\delta\widehat{X})^n-\widehat{X}^n}{n}\;.
\end{aligned}
$$

Since, in general, $\widehat{X}$ and $\delta\widehat{X}$ do not commute, we have to keep track of all possible positions of $\delta\widehat{X}$. To first order,

$$
(\widehat{X}+\delta\widehat{X})^n=\widehat{X}^n+\widehat{X}^{n-1}\delta\widehat{X}+\widehat{X}^{n-2}\delta\widehat{X}\,\widehat{X}+\cdots+\delta\widehat{X}\,\widehat{X}^{n-1}+\mathcal{O}(\delta\widehat{X}^2)\;.
$$

Therefore,

$$
\begin{aligned}
&\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\\
&=-\sum_{n=1}^{\infty}\frac{\widehat{X}^{n-1}\delta\widehat{X}+\widehat{X}^{n-2}\delta\widehat{X}\,\widehat{X}+\cdots+\delta\widehat{X}\,\widehat{X}^{n-1}}{n}+\mathcal{O}(\delta\widehat{X}^2)\;.
\end{aligned}
$$

Now we plugging this in the variation of the entropy and using the cyclic property of the trace we find that all terms contribute in the same way:

$$
\begin{aligned}
&\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}\\
&=-\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X})\delta\widehat{X}\sum_{n=1}^{\infty}\widehat{X}^{n-1}\right]+\mathcal{O}(\delta\widehat{X}^2)\;.
\end{aligned}
$$

Since the sum reduces to a geometric series:

$$
\sum_{n=1}^{\infty}\widehat{X}^{n-1}=(\widehat{\mathrm{I}}-\widehat{X})^{-1}\;, \tag{9}
$$

we obtain

$$
\begin{aligned}
&\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}\\
&=-\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X})\delta\widehat{X}(\widehat{\mathrm{I}}-\widehat{X})^{-1}\right]+\mathcal{O}(\delta\widehat{X}^2)\\
&=-\operatorname{Tr}(\delta\widehat{X})+\mathcal{O}(\delta\widehat{X}^2)\;.
\end{aligned}
$$

Therefore the first-order variation of the functional is

$$
F[\,\widehat{X}+\delta\widehat{X}\,]-F[\,\widehat{X}\,]=\operatorname{Tr}\left[\delta\widehat{X}\,\ln(\widehat{\mathrm{I}}-\widehat{X})\right]+\operatorname{Tr}(\delta\widehat{X})+c\operatorname{Tr}(\delta\widehat{X})+\lambda\operatorname{Tr}(\delta\widehat{X}\,\widehat{A})+\mathcal{O}(\delta\widehat{X}^2)\;,
$$

or equivalently:

$$F[\,\widehat{X}+\delta\widehat{X}\,]-F[\,\widehat{X}\,]=\mathrm{Tr}\left\{\delta\widehat{X}\left[\ln(\widehat{\mathrm{I}}-\widehat{X})+\widehat{\mathrm{I}}+c\widehat{\mathrm{I}}+\lambda\widehat{A}\right]\right\}+\mathcal{O}(\delta\widehat{X}^2)\;. \tag{10}$$

We can now rewrite this result in terms of $\widehat{\rho}$ for which

$$\widehat{\rho}=\widehat{\mathrm{I}}-\widehat{X}\implies\delta\widehat{\rho}=-\delta\widehat{X}\;.$$

Then:

$$F[\,\widehat{\rho}+\delta\widehat{\rho}\,]-F[\,\widehat{\rho}\,]=-\mathrm{Tr}\left\{\delta\widehat{\rho}\left[\ln\widehat{\rho}+\widehat{\mathrm{I}}+c\widehat{\mathrm{I}}+\lambda\widehat{A}\right]\right\}+\mathcal{O}(\delta\widehat{\rho}^2)\;. \tag{11}$$

At the stationary point, the first-order variation must vanish for arbitrary Hermitian variations $\delta\widehat{\rho}$. Therefore,

$$\ln\widehat{\rho}+\widehat{\mathrm{I}}+c\widehat{\mathrm{I}}+\lambda\widehat{A}=0\;,$$

which implies

$$\ln\widehat{\rho}=-(1+c)\widehat{\mathrm{I}}-\lambda\widehat{A}\;.$$

Exponentiating both sides, and using the fact that the identity operator commutes with $\widehat{A}$, we obtain

$$\widehat{\rho}=\mathrm{e}^{-(1+c)\widehat{\mathrm{I}}-\lambda\widehat{A}}=\mathrm{e}^{-(1+c)}\mathrm{e}^{-\lambda\widehat{A}}\;. \tag{12}$$

The constant $\mathrm{e}^{-(1+c)}$ is fixed by the normalization condition $\mathrm{Tr}\widehat{\rho}=1$. Therefore,

$$1=\mathrm{Tr}\widehat{\rho}=\mathrm{e}^{-(1+c)}\mathrm{Tr}\left(\mathrm{e}^{-\lambda\widehat{A}}\right)\;.$$

Defining the partition function as

$$Z\equiv\mathrm{Tr}\left(\mathrm{e}^{-\lambda\widehat{A}}\right)\;, \tag{13}$$

and identifying $Z=\mathrm{e}^{-(1+c)}$ we finally obtain:

$$\widehat{\rho}=\frac{1}{Z}\mathrm{e}^{-\lambda\widehat{A}}\;. \tag{14}$$

The Lagrange multiplier λ is fixed by the condition

$$\langle\widehat{A}\rangle=\mathrm{Tr}(\widehat{\rho}\,\widehat{A})\;. \tag{15}$$

In terms of the partition function, this relation can also be written as

$$\langle\widehat{A}\rangle=-\frac{\partial\ln Z}{\partial\lambda}\;. \tag{16}$$

Finally, since the von Neumann entropy is a concave functional of $\widehat{\rho}$ and the constraints are linear in $\widehat{\rho}$, the stationary point found above is the maximum of the entropy under the imposed constraints.

Now lets specialize to the canonical case. Note that, as long as they are commuting, we can simply replace $\widehat{A}$ with any set of operators we want each of one with its own Lagrange multiplier:

$$\widehat{\rho}=\frac{1}{Z}\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}\;,\qquad Z=\mathrm{Tr}\left(\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}\right)\;.$$

Then is easy to prove that:

$$\langle\widehat{P}^\nu\rangle=-\frac{\partial\ln Z}{\partial\beta_\nu}\;,\qquad\langle\widehat{Q}\rangle=\frac{\partial\ln Z}{\partial\zeta}$$

I. EXERCISE 2

For a relativistic system, the possible global-equilibrium configurations are richer than in the non-relativistic case. Besides homogeneous equilibrium, one can have equilibrium states with non-vanishing thermal vorticity. In the simplest situations, these configurations can be organized into two basic classes: global equilibrium with rotation and global equilibrium with acceleration. The first one has a direct classical analogue, namely a rigidly rotating body in thermal equilibrium. The second one is intrinsically relativistic and is associated with Lorentz boosts.

Let us start from the operator maximizing the entropy,

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \zeta \right) \right] \, . \tag{17}$$

The condition for global equilibrium is that the operator be independent of the integration hypersurface Σ. This means that the current appearing in the exponent must be conserved. Therefore, we require

$$\nabla_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \zeta \right) = 0 \, . \tag{18}$$

Expanding the derivative, we obtain

$$\nabla_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \zeta \right) = \beta_\nu \nabla_\mu \widehat{T}^{\mu\nu} + \widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu - \zeta \nabla_\mu \widehat{j}^\mu - \widehat{j}^\mu \nabla_\mu \zeta \, .$$

Using the conservation equations

$$\nabla_\mu \widehat{T}^{\mu\nu} = 0 \, , \qquad \nabla_\mu \widehat{j}^\mu = 0 \, ,$$

the condition reduces to

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta = 0 \, . \tag{19}$$

Since the stress-energy tensor and the conserved current are independent hydrodynamic operators, the global-equilibrium condition requires separately

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu = 0 \, , \qquad \widehat{j}^\mu \nabla_\mu \zeta = 0 \, .$$

The second equation implies that the reduced chemical potential ζ must be constant:

$$\nabla_\mu \zeta = 0 \, .$$

For the four-temperature, we use the symmetry of the stress-energy tensor. Since $\widehat{T}^{\mu\nu} = \widehat{T}^{\nu\mu}$, we can write

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu = \frac{1}{2} \widehat{T}^{\mu\nu} \left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu \right) + \frac{1}{2} \widehat{T}^{\mu\nu} \left(\nabla_\mu \beta_\nu - \nabla_\nu \beta_\mu \right) \, .$$

The second term vanishes because it is the contraction of a symmetric tensor with an antisymmetric tensor. Thus,

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu = \frac{1}{2} \widehat{T}^{\mu\nu} \left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu \right) \, .$$

For this expression to vanish in general, the four-temperature must satisfy

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \, . \tag{20}$$

This is the Killing equation. Therefore, at global equilibrium, the four-temperature β^μ must be a Killing vector.

A Killing vector has a clear geometrical meaning: it is the generator of an isometry of spacetime. In other words, the flow generated by a Killing vector leaves the metric invariant. This can be written as

$$\mathfrak{L}_\beta g_{\mu\nu} = 0 \, ,$$

where $\mathfrak{L}_\beta$ denotes the Lie derivative along β^μ. Therefore, the field lines of β^μ are the curves along which the geometry is unchanged. This is the geometrical reason why a thermal state can be in equilibrium even when β^μ depends explicitly on the spacetime point: what matters is not ordinary homogeneity, but stationarity with respect to the flow generated by β^μ. The fact that β^μ is a Killing vector is not yet enough to guarantee a thermodynamic interpretation. In order to describe a physical thermal state, the four-temperature has to be timelike and future-directed:

$$\beta^2 > 0 \, , \qquad \beta^0 > 0 \, .$$

Only in the region where these conditions hold can we write

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)} \, , \qquad T(x) = \frac{1}{\sqrt{\beta^2(x)}} \, , \qquad u^\mu(x) = \frac{\beta^\mu(x)}{\sqrt{\beta^2(x)}} \, . \tag{21}$$

Here u^μ is the local four-velocity of the fluid and T is the proper temperature, namely the temperature measured by a thermometer comoving with the fluid element.

An important consequence of the Killing equation is that the proper temperature is constant along the flow lines of β^μ. To see this, we contract the Killing equation (20) with $\beta^\mu\beta^\nu$:

$$\beta^\mu\beta^\nu \left(\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu\right) = 0 \, .$$

The two terms are equal, and therefore

$$2\beta^\mu\beta^\nu\nabla_\mu\beta_\nu = 0 \, .$$

On the other hand,

$$\beta^\mu\nabla_\mu(\beta^2) = \beta^\mu\nabla_\mu(\beta^\nu\beta_\nu) = 2\beta^\mu\beta^\nu\nabla_\mu\beta_\nu \, .$$

Thus

$$\beta^\mu\nabla_\mu(\beta^2) = 0 \, . \tag{22}$$

Using $T = 1/\sqrt{\beta^2}$, equation (22) implies

$$\beta^\mu\nabla_\mu T = 0 \, .$$

Since u^μ is proportional to β^μ, we also have

$$u^\mu\nabla_\mu T = 0 \, .$$

Therefore, a comoving observer measures a constant temperature along its own worldline. However, different flow lines can have different values of the proper temperature. This point is crucial for understanding both rotating and accelerated equilibrium.

We now solve the equation for the particular case of Minkowski space-time. First, for Minkowski, the Christoffel are vanishing and the covariant derivative is simply reduced to the partial one. We then must solve the following set of equations:

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0 \, . \tag{23}$$

The solution of this equation is quite simple. Take the derivative with respect to ∂_λ:

$$\partial_\lambda\partial_\mu\beta_\nu + \partial_\lambda\partial_\nu\beta_\mu = 0 \, .$$

Define the tensor $B_{\lambda\mu\nu} \equiv \partial_\lambda\partial_\mu\beta_\nu$. Due to the Killing equation (23) the tensor is antisymmetric in the last two indeces whereas due to the commutativity of partial derivatives it is symmetric in the first two:

$$B_{\lambda\mu\nu} = B_{\mu\lambda\nu} \, , \qquad B_{\lambda\mu\nu} = -B_{\lambda\nu\mu} \, . \tag{24}$$

But this implies:

$$B_{\lambda\mu\nu} = -B_{\lambda\nu\mu} = -B_{\nu\lambda\mu} = B_{\nu\mu\lambda} = B_{\mu\nu\lambda} = -B_{\mu\lambda\nu} = -B_{\lambda\mu\nu} \, ,$$

i.e. $B_{\lambda\mu\nu} = -B_{\lambda\mu\nu} = 0$:

$$\partial_\lambda\partial_\mu\beta_\nu = 0 \, . \tag{25}$$

The four-temperature cannot depend more than linearly from x:

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \ , \tag{26}$$

where both b and ϖ are constant. In particular the latter has to be antisymmetric due to the Killing equation (23):

$$\partial_\mu \beta_\nu = \varpi_{\mu\nu} = -\partial_{\nu\mu} = -\varpi_{\nu\mu} \ ,$$

and one can define the thermal vorticity as:

$$\varpi_{\mu\nu} = \frac{1}{2}\left(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu\right) \ . \tag{27}$$

Then in Minkowski spacetime the general solution of the Killing equation is:

$$\beta_\mu(x) = b_\mu + \varpi_{\mu\nu} x^\nu \ , \tag{28}$$

The four constants contained in b_μ are associated with translations, while the six independent constants contained in $\varpi_{\mu\nu}$ are associated with Lorentz transformations, namely rotations and boosts. Thus the most general Killing vector in Minkowski spacetime is determined by ten constants, corresponding to the ten generators of the Poincare group.

We can show how the thermal vorticity encodes both the effects of acceleration and rotation. Indeed any antisymmetric rank two tensor can be decomposed in two space-like four-vectors. A generic space-like four vector is orthogonal to a given time-like four-vector. We have at disposal the four-velocity so we can define:

$$\alpha^\mu = \varpi^{\mu\nu} u_\nu \ , \qquad w^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \varpi_{\rho\sigma} \ . \tag{29}$$

Due to the antisymmetricity of ϖ and ϵ it follows:

$$\alpha^\mu u_\mu = 0 \ , \quad w^\mu u_\mu = 0 \ , \tag{30}$$

and:

$$\varpi_{\mu\nu} = \alpha_\mu u_\nu - \alpha_\mu u_\nu + \epsilon_{\mu\nu\rho\sigma} w^\rho u^\sigma \ . \tag{31}$$

Now, component wise, we have:

$$\varpi_{0i} = \alpha^i \ , \quad \varpi_{ij} = \epsilon_{ijk} w^k \ ,$$

that is:

$$\varpi_{\mu\nu} \sim \begin{pmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ -\alpha_x & 0 & w_z & -w_y \\ -\alpha_y & -w_z & 0 & w_x \\ -\alpha_z & w_y & -w_x & 0 \end{pmatrix} \ . \tag{32}$$

Now we can clearly see how α and w are directly related with the four-acceleration vector A and the kinetic vorticity ω defined as:

$$A^\mu = u^\nu \partial_\nu u^\mu \ , \qquad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \ . \tag{33}$$

We start from the four-acceleration. Given that $u_\mu = T\beta_\mu$ we have:

$$A^\mu = u^\nu \partial_\nu (T\beta^\mu) = \beta^\mu u^\nu \partial_\nu T + u^\nu T \partial_\nu \beta^\mu \ .$$

The first term vanishes given that the temperature is constant along the flow lines while for the second one we can replace the gradient with the vorticity using the solution (28):

$$A^\mu = T u^\nu \varpi^\mu_\nu = T\alpha^\mu \ ,$$

so that:

$$\alpha^\mu = \frac{A^\mu}{T} \ . \tag{34}$$

Now lets consider w^μ. We get:

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho(T\beta_\sigma) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\left(\beta_\sigma\partial_\rho T + T\partial_\rho\beta_\sigma\right) \ .$$

Being $u_\nu = T\beta_\nu$ the first piece is proportional to $\epsilon^{\mu\nu\rho\sigma}\beta_\nu\beta_\rho = 0$ which is vanishing due to antisimmetricity. The second term instead is reletated with the thermal vorticity (27):

$$\partial_\rho\beta_\sigma = \frac{1}{2}(\partial_\rho\beta_\sigma - \partial_\sigma\beta_\rho) + \frac{1}{2}(\partial_\rho\beta_\sigma + \partial_\sigma\beta_\rho) = \varpi_{\rho\sigma} \ ,$$

where the symmetric part vanishes due to (23), so that:

$$\omega^\mu = \frac{T}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\varpi_{\rho\sigma} = Tw^\mu \ ,$$

that is:

$$w^\mu = \frac{\omega^\mu}{T} \ . \tag{35}$$

We now return to the particular case of Minkowski spacetime. From (28), one sees that when $\varpi_{\mu\nu} \neq 0$, the four-temperature depends explicitly on the spacetime coordinate x^ν. Therefore, in a generic inertial frame, the equilibrium state is not homogeneous and, in some cases, it is not even time independent with respect to the inertial time coordinate. Nevertheless, it is still a global-equilibrium state because it is stationary with respect to the flow generated by the Killing vector β^μ.

In the following, we discuss two important examples: global equilibrium with rotation and global equilibrium with acceleration.