Hydrodynamic as an effective field theory

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INTRODUCTION

These notes are meant to accompany the lectures given by Prof. Francesco Becattini for the BIP (Blended Intensive Programme) Relativistic Hydrodynamics, held in Timișoara, Romania, from 25/05/2026 to 29/05/2026. The lectures provide a crash course in non-equilibrium statistical mechanics and its applications to hydrodynamics. In particular, we show how the macroscopic laws of hydrodynamics can be derived from the most fundamental theoretical framework currently available, namely quantum field theory. In this sense, relativistic hydrodynamics provides a direct bridge between the microscopic and macroscopic descriptions of physical systems.

I. NON-EQUILIBRIUM RELATIVISTIC STATISTICAL MECHANICS

In statistical mechanics we often want to describe a macroscopic system whose microscopic state is not known. From the quantum-mechanical point of view, this means that we do not know which vector in the Hilbert space represents the actual state of the system. We therefore describe the system as a statistical mixture of possible microscopic states: the number w_i is the probability that the system is in the state $|\psi_i\rangle$.

The state is then described by a matrix, built out of the microscopic states, called the *density operator* or *statistical operator*:

$$\widehat{\rho} = \sum_{i=1}^{N} w_i |\psi_i\rangle\langle\psi_i| \ . \tag{1}$$

Since the coefficients w_i are classical probabilities, the density operator is normalized accordingly:

$$\sum_{i=1}^{N} w_i = 1 \implies \mathrm{Tr}\widehat{\rho} = 1 \ , \tag{2}$$

where the trace Tr denotes the sum over a complete set of states. The amount of missing information about the microscopic state of the system is quantified by the Von Neumann, or information, entropy:

$$S = -\mathrm{Tr}(\widehat{\rho}\ln\widehat{\rho}) = -\sum_{i=1}^{N} w_i \ln w_i \ . \tag{3}$$

For a genuine statistical mixture the entropy is positive, while its minimum value, $S = 0$, is reached only when one state has probability one and all the others have probability zero. Because of the normalization condition, this can happen for one state only. In that case,

$$\widehat{\rho} = |\psi_i\rangle\langle\psi_i| \equiv \widehat{\mathrm{P}}_{\psi_i} \ , \tag{4}$$

so the density operator reduces to the projector onto the microscopic state $|\psi_i\rangle$. This is the case of maximal information: the precise microscopic state of the system is known. Given that a statistical operator is built in terms of the states, (1), it then follows that in Heisenberg picture, where the states are fixed, the density operator must be stationary, i.e. it must solve:

$$\frac{\mathrm{d}\widehat{\rho}}{\mathrm{d}t} = 0 \ , \tag{5}$$

also known as Liouville equation.

The expectation value of an observable $\widehat{O}$ in a statistical state is obtained by averaging the quantum expectation value over all possible microscopic states, weighted by their probabilities:

$$\langle\widehat{O}\rangle = \sum_{i=1}^{N} w_i \langle\psi_i|\widehat{O}|\psi_i\rangle = \mathrm{Tr}\left(\widehat{\rho}\,\widehat{O}\right) \ . \tag{6}$$

We now want to connect this statistical description with a thermodynamic one. Consider a macroscopic system made of a very large number N of microscopic constituents, each of them in a state $|\phi_k\rangle$. A possible microscopic state of the full system may be written schematically as

$$|\psi_i\rangle = \bigotimes_{k=1}^{N} |\phi_k\rangle \ .$$

In principle, each possible combination of the states $|\phi_k\rangle$ gives a different microscopic state of the whole system. For macroscopic N, however, this information is completely inaccessible in practice.

Let us assume that the system is in contact with a much larger environment, with which it can exchange energy, momentum and particles. When the system equilibrates with the environment, the total entropy is maximized, in agreement with the second law of thermodynamics. In a thermodynamic description, the state of the system is characterized only by a finite set of macroscopic observables $\widehat{A}_i$, whose expectation values are fixed. One could impose exact values of the conserved quantities, as in a microcanonical description, but here we are interested in the situation where the system can exchange these quantities with the surroundings. Therefore they are fixed only on average.

The thermodynamic state is then defined as the statistical state that maximizes the entropy subject to the constraints of fixed average observables and normalized density operator. In other words, one maximizes

$$F[\,\widehat{\rho}\,] \equiv -\mathrm{Tr}(\widehat{\rho}\ln\widehat{\rho}) + c\,(\mathrm{Tr}\widehat{\rho} - 1) + \sum_i \lambda_i \left(\mathrm{Tr}(\widehat{\rho}\,\widehat{A}_i) - \langle\widehat{A}_i\rangle\right) \ . \tag{7}$$

The quantities λ_i and c are Lagrange multipliers. They implement, respectively, the constraints on the average values of the observables and on the normalization of the density operator. The general solution is

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\sum_i \lambda_i \widehat{A}_i\right) \ . \tag{8}$$

A familiar example is obtained when the relevant macroscopic quantities are the energy and the total number of particles. This gives the usual *grand-canonical density operator*:

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\beta\widehat{H} + \zeta\widehat{Q}\right) \ , \tag{9}$$

where the two Lagrange multipliers are identified with the inverse temperature $\beta = 1/T$ and the reduced chemical potential $\zeta = \mu/T$. Their physical interpretation follows by comparing the maximum-entropy construction with the ordinary thermodynamic relation between entropy, energy and conserved charge: the multiplier conjugate to the energy is the inverse temperature, while the multiplier conjugate to the charge is the chemical potential divided by the temperature.

The operator in (9) is suitable for a non-relativistic formulation, but it is not written in a manifestly covariant form: the Hamiltonian $\widehat{H}$ is the time component of the four-momentum and is not a Lorentz scalar. To construct the relativistic density operator it is better to use a covariant formulation from the beginning. This is also the most natural starting point for the later extension to non-equilibrium situations.

A macroscopic system is effectively described in terms of a set of *macroscopic observables*. In order to be associated with the system as a whole, these observables must be *global*: they must be defined over the entire system. In relativity, global observables are naturally written as integrals of conserved local currents. If a current $J^{\mu\mu_1\cdots}$ is conserved,

$$\partial_\mu J^{\mu\mu_1\cdots} = 0 \ , \tag{10}$$

then its integral over a spacelike hypersurface is independent of the choice of hypersurface:

$$\int_\Sigma \mathrm{d}\Sigma_\mu \, J^{\mu\mu_1\cdots} \equiv Q^{\mu_1\cdots} \ \forall\,\Sigma \ . \tag{11}$$

This is the covariant version of the statement that a global charge is obtained by integrating the corresponding density over space. If the current is conserved, the charge is independent of the time parameter in any inertial reference frame.

The two currents that will be relevant for us are the stress-energy tensor and the baryon current. In a quantum system they are promoted to operators on the Hilbert space and satisfy the conservation equations

$$\partial_\mu \widehat{T}^{\mu\nu} = 0 \,, \quad \partial_\mu \widehat{j}^{\mu}_{\rm B} = 0 \,. \tag{12}$$

Therefore their integrals over an arbitrary spacelike hypersurface are well-defined and correspond, respectively, to the total four-momentum and the total baryon number:

$$\widehat{P}^\nu = \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu} \,, \qquad \widehat{Q}_{\rm B} = \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{j}^{\mu}_{\rm B} \,. \tag{13}$$

From the stress-energy tensor one can also construct the conserved Lorentz generators, which include angular momentum and boosts:

$$\widehat{J}^{\nu\lambda} = \int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\lambda} x^\nu - \widehat{T}^{\mu\nu} x^\lambda \right) \,. \tag{14}$$

Note that the integrand has vanishing divergence due to (12). Hence, for symmetric stress-energy tensor, the conservation of the angular momentum does not introduce another Lagrange multiplier in the maximization and thus no other independent thermodynamic field appears.

We can now repeat the maximum-entropy construction in a covariant form. The entropy is maximized at fixed average values of the conserved currents on a spacelike hypersurface $\Sigma(\tau)$:

$$F[\widehat{\rho}, \Sigma] = -\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho}) + c\,(\mathrm{Tr}\widehat{\rho} - 1) + \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left\{ \beta_\nu \left[\mathrm{Tr}\left(\widehat{\rho} \widehat{T}^{\mu\nu} \right) - \langle \widehat{T}^{\mu\nu} \rangle \right] - \zeta \left[\mathrm{Tr}\left(\widehat{\rho} \widehat{j}^{\mu}_{\rm B} \right) - \langle \widehat{j}^{\mu}_{\rm B} \rangle \right] \right\} \,. \tag{15}$$

The constraint on the total angular momentum is redundant once the stress-energy tensor is assumed to be symmetric, because the Lorentz generators are then already determined by the stress-energy tensor.

The corresponding solution is

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^{\mu}_{\rm B} \zeta \right) \right] \,, \tag{16}$$

where the matching with the thermodynamic equilibrium relations tells us that β_μ and ζ have now the physical meaning of four-temperature and reduced chemical potential:

$$\beta_\mu = \frac{u_\mu}{T} \,, \qquad \zeta = \frac{\mu}{T} \,. \tag{17}$$

The vector field u^μ represents the four-velocity of the fluid, $u^2 = 1$, while T is the proper temperature, i.e. the temperature measured by a thermomether comoving with the system. Note that in this framework the fundamental quantity is the four-temperature whereas the four-velocity is a derived quantity:

$$u_\mu(x) \equiv \frac{\beta_\mu(x)}{\sqrt{\beta^2(x)}} \,. \tag{18}$$

This can be derived from the definition of entropy:

$$\begin{aligned} S = -\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho}) &= -\mathrm{Tr}\left[\widehat{\rho} \left(-\ln Z + \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu (\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^{\mu} \zeta) \right) \right] \\ &= \ln + \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\langle \widehat{T}^{\mu\nu} \rangle \beta_\nu - \langle \widehat{j}^{\mu} \rangle \zeta \right) \,, \end{aligned}$$

and assuming that the thermodynamic field are constant implies:

$$S = \ln Z_{\rm LE} + \beta_\nu \langle \widehat{P}^\nu \rangle - \zeta \langle \widehat{Q} \rangle \,.$$

Comparing with the classical expression:

$$TS = \ln Z + u \cdot P - \mu Q \,, \tag{19}$$

then leads to the identification (17).

At this stage the operator (16) seems to depend on the hypersurface Σ. In true thermodynamic equilibrium, however, the statistical operator must be independent of the hypersurface used in the maximization. This is equivalent to requiring the integrand to be divergenceless:

$$\nabla_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) = \widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta = 0 \, .$$

For this condition to hold independently of the particular state of the system, the four-temperature field must be a Killing vector and the reduced chemical potential to be covariantly conserved. If we consider the case of Minkowski space-time in cartesian coordinates then $\nabla \to \partial$ and the equilibrium correction turns out to be (see exercises):

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu \;=\; 0 \, , \implies \beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \, , \tag{20}$$

$$\partial_\mu \zeta \;=\; 0 \, , \implies \zeta = \text{const.} \tag{21}$$

where b_μ is a constant four-vector, while $\varpi_{\mu\nu}$ is a constant antisymmetric tensor called *thermal vorticity*.

Thus, global equilibrium can be characterized geometrically: the four-temperature is a Killing vector, and the reduced chemical potential is constant. Using these conditions in (16), the dependence on the hypersurface disappears, as expected:

$$b_\nu \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu} - \frac{1}{2} \varpi_{\nu\lambda} \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\lambda} x^\nu - \widehat{T}^{\mu\nu} x^\lambda \right) - \zeta \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \, \widehat{j}^\mu_{\mathrm{B}} = b \cdot \widehat{P} - \frac{1}{2} \varpi : \widehat{J} - \zeta \widehat{Q}_{\mathrm{B}} \, .$$

We then obtain the most general global-equilibrium operator:

$$\widehat{\rho}_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} \exp \left(-b \cdot \widehat{P} + \frac{1}{2} \varpi : \widehat{J} + \zeta \widehat{Q} \right) \, . \tag{22}$$

In general this state is neither homogeneous nor isotropic.

The usual grand-canonical result is recovered in the special case of vanishing thermal vorticity. In that case,

$$\widehat{\rho}_{\mathrm{GE}} = \frac{1}{Z} \exp \left(-\beta \cdot \widehat{P} + \zeta \widehat{Q} \right) \tag{23}$$

which reduces to (9) in the local rest frame, since $\widehat{P}^0 = \widehat{H}$. This state is usually called *homogeneous global equilibrium*. It is invariant under spatial translations and rotations. However, it is not the most general equilibrium state allowed by relativity. A non-vanishing thermal vorticity allows global equilibrium configurations that are inhomogeneous or anisotropic. In such cases, the temperature measured by comoving observers can vary in space, as happens for instance in rotating or accelerated equilibrium. This is the relativistic content of the Tolman–Ehrenfest law.

A. Non-equilibrium density operator

The previous discussion led to the construction of the global equilibrium density operator. We now want to move one step further and describe a system which is not in global equilibrium, but which is still close enough to equilibrium that a thermodynamic language remains meaningful locally.

Local equilibrium, despite its name, is not true thermodynamic equilibrium. It is a special kind of non-equilibrium state in which thermodynamic variables can be assigned locally. Classically, one may think of a system whose temperature, chemical potential and velocity vary from point to point. The system as a whole is not in equilibrium, but if we look at a sufficiently small region, these quantities are approximately constant and a local thermodynamic description can be used.

This picture relies on a separation of scales. On the one hand, there is the microscopic scale over which the local state is established, for instance the mean free path or the relaxation scale. On the other hand, there is the macroscopic scale over which the thermodynamic fields vary appreciably. Local equilibrium is meaningful only when these two scales are well separated. In that case, each small fluid cell can be described by thermodynamic variables, even though the system as a whole is evolving out of equilibrium.

The question is how to formulate this idea covariantly. Away from global equilibrium, it is no longer useful to characterize the state only in terms of global conserved quantities. For example, although the total energy of an isolated system is conserved, the local energy density and the local momentum density vary in space and time. What remains naturally defined are the local conserved currents.

Therefore, on a point x of a given space-like hypersurface Σ, with future-directed unit normal vector $\hat{n}$, one introduces local thermodynamic fields $\beta(x)$ and $\zeta(x)$ and defines an operator $\widehat{\rho}_{\rm LE}$ which reproduces the local state on that hypersurface:

$$\begin{aligned} \hat{n}_\mu \mathrm{Tr}\left(\widehat{\rho}_{\rm LE}\,\widehat{T}^{\mu\nu}(x)\right)[\beta,\zeta,\hat{n}] &\equiv \hat{n}_\mu \langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm LE} = \hat{n}_\mu \langle \widehat{T}^{\mu\nu}(x)\rangle \ , \\ \hat{n}_\mu \mathrm{Tr}\left(\widehat{\rho}_{\rm LE}\,\widehat{j}^{\mu}(x)\right)[\beta,\zeta,\hat{n}] &\equiv \hat{n}_\mu \langle \widehat{j}^{\mu}(x)\rangle_{\rm LE} = \hat{n}_\mu \langle \widehat{j}^{\mu}(x)\rangle \ . \end{aligned} \tag{24}$$

This operator is obtained by maximizing the entropy, in close analogy with the construction of the global equilibrium operator, but now with the local constraints above. i.e. the local equilibrium opeator is the operator maximizing the following functional:

$$\begin{aligned} F[\,\widehat{\rho},\Sigma(\tau)\,] = &-\mathrm{Tr}(\widehat{\rho}_{\rm LE}\ln\widehat{\rho}_{\rm LE}) + c\,[\mathrm{Tr}(\widehat{\rho}_{\rm LE})-1] \\ &+\int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left\{\beta_\mu\left[\mathrm{Tr}(\widehat{\rho}_{\rm LE}\,\widehat{T}^{\mu\nu}) - \langle \widehat{T}^{\mu\nu}\rangle_{\rm LE}\right] - \zeta\left[\mathrm{Tr}(\widehat{\rho}_{\rm LE}\,\widehat{j}^{\mu}) - \langle \widehat{j}^{\mu}\rangle_{\rm LE}\right]\right\} \ , \end{aligned} \tag{25}$$

which is formally the same as (15). The general solution is given by:

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^{\mu}\zeta\right)\right] \ . \tag{26}$$

This is usually called the *local equilibrium operator*. The quantities on the right-hand side of (24) are the true non-equilibrium expectation values, whereas the expressions on the left-hand side are computed with the local equilibrium operator. The square brackets emphasize that these local-equilibrium expectation values are functionals of the thermodynamic fields and of the chosen hypersurface.

The reason why the matching conditions involve the projections along $\hat{n}_\mu$ is not primarily that information cannot propagate along a space-like hypersurface. Rather, it is a consequence of the maximum-entropy construction on a chosen hypersurface: the local constraints are imposed on the densities measured by the observer normal to Σ. In this sense, $\widehat{\rho}_{\rm LE}$ is a pseudo-equilibrium operator adapted to the hypersurface Σ. It reproduces the local densities fixed by the matching conditions, but it does not describe a stationary equilibrium state.

The name "local equilibrium density operator" can therefore be slightly misleading. The operator $\widehat{\rho}_{\rm LE}$ is constructed to reproduce the local state on a given hypersurface, but it is not stationary in the Heisenberg picture. In particular, it does not satisfy the Liouville equation (5). The reason is that, out of global equilibrium, the fields β and ζ do not satisfy the global-equilibrium conditions, and the operator explicitly depends on the hypersurface on which the entropy maximization is performed.

Let us now assume that the system is in local thermodynamic equilibrium on an initial hypersurface $\Sigma_0 \equiv \Sigma(\tau_0)$. The operator

$$\widehat{\rho} \equiv \widehat{\rho}_{\rm LE}(\tau_0) = \frac{1}{Z}\exp\left[-\int_{\Sigma_0} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^{\mu}\zeta\right)\right] \tag{27}$$

then describes the state of the system on Σ_0. Since it is defined once and for all from the initial data, it is the statistical operator used to compute non-equilibrium expectation values. In the Heisenberg picture, the state is fixed, while the operators carry the time dependence.

To express the same operator in terms of fields on a later hypersurface $\Sigma(\tau)$, we apply Gauss' theorem to the space-time region Ω enclosed between Σ_0 and $\Sigma(\tau)$. Using the conservation equations for $\widehat{T}^{\mu\nu}$ and $\widehat{j}^{\mu}$, one obtains

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu (\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^{\mu}\zeta) + \int_\Omega \mathrm{d}\Omega \left(\widehat{T}^{\mu\nu}\partial_\mu\beta_\nu - \widehat{j}^{\mu}\partial_\mu\zeta\right)\right] \ . \tag{28}$$

The first term has the same structure as the local equilibrium operator on the final hypersurface. The second term depends on the space-time history between Σ_0 and $\Sigma(\tau)$ and is controlled by gradients of the thermodynamic fields. It is this term which contains the dissipative departure from local equilibrium.

B. Linear response theory

Once the non-equilibrium state has been defined, the main goal is to compute expectation values of observables in that state. This is generally difficult, especially when dissipation is important, because dissipative effects depend on the microscopic interactions of the system.

However, if the system is close to local equilibrium, the gradients of the thermodynamic fields are small on the relevant microscopic scales. This makes it possible to treat the non-equilibrium part perturbatively. Starting from the non-equilibrium operator above, we write

$$\widehat{\rho} = \frac{1}{Z} \exp\left(\widehat{\mathcal{E}} + \widehat{\mathcal{D}}\right) , \qquad Z = \mathrm{Tr}\left[\exp\left(\widehat{\mathcal{E}} + \widehat{\mathcal{D}}\right)\right] , \tag{29}$$

where $\widehat{\mathcal{E}}$ is the local-equilibrium part and $\widehat{\mathcal{D}}$ is the dissipative part:

$$\begin{aligned} \widehat{\mathcal{E}} &= -\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \zeta\right) , \\ \widehat{\mathcal{D}} &= \int_\Omega \mathrm{d}\Omega \left(\widehat{T}^{\mu\nu}\partial_\mu\beta_\nu - \widehat{j}^\mu \partial_\mu \zeta\right) . \end{aligned} \tag{30}$$

The operator $\widehat{\mathcal{D}}$ is proportional to gradients of the thermodynamic fields. It is therefore natural, in the regime of slowly varying fields, to regard it as a small correction to $\widehat{\mathcal{E}}$ and expand the density operator in powers of $\widehat{\mathcal{D}}$.

The useful identity is the Kubo identity, which follows from the Baker–Campbell–Hausdorff formula:

$$\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}} = \mathrm{e}^{\widehat{\mathcal{E}}} + \left(\int_0^1 \mathrm{d}z\, \mathrm{e}^{z(\widehat{\mathcal{E}}+\widehat{\mathcal{D}})}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}} . \tag{31}$$

At first order in $\widehat{\mathcal{D}}$ this becomes

$$\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}} \simeq \mathrm{e}^{\widehat{\mathcal{E}}} + \left(\int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}} . \tag{32}$$

The same expansion can be applied to the normalization factor:

$$\begin{aligned} Z = \mathrm{Tr}\left(\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}}\right) &\simeq \mathrm{Tr}\left[\mathrm{e}^{\widehat{\mathcal{E}}} + \left(\int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}}\right] \\ &= \mathrm{Tr}(\mathrm{e}^{\widehat{\mathcal{E}}}) + \mathrm{Tr}\left[\left(\int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}}\right] \\ &\equiv Z_{\mathrm{LE}} + Z_{\mathrm{LE}} \int_0^1 \mathrm{d}z\, \mathrm{Tr}\left[\frac{\mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathrm{LE}}}\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right] . \end{aligned}$$

Here $Z_{\mathrm{LE}} = \mathrm{Tr}(\exp(\widehat{\mathcal{E}}))$, and the operator $\exp(\widehat{\mathcal{E}})/Z_{\mathrm{LE}}$ is precisely the local equilibrium operator. Therefore

$$\mathrm{Tr}\left[\frac{\mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathrm{LE}}}\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right] \equiv \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\mathrm{LE}} .$$

The partition function of the full operator is then approximated by

$$Z \simeq Z_{\mathrm{LE}} \left(1 + \int_0^1 \mathrm{d}z\, \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\mathrm{LE}}\right) . \tag{33}$$

Using this result in the expression for $\widehat{\rho}$ and expanding the denominator to first order gives

$$\begin{aligned} \widehat{\rho} &\simeq \frac{\mathrm{e}^{\widehat{\mathcal{E}}} + \left(\int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathrm{LE}} \left(1 + \int_0^1 \mathrm{d}z\, \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\mathrm{LE}}\right)} \\ &\simeq \left[\mathrm{e}^{\widehat{\mathcal{E}}} + \left(\int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}}\right] \frac{1}{Z_{\mathrm{LE}}} \left[1 - \int_0^1 \mathrm{d}z\, \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\mathrm{LE}}\right] \\ &\simeq \frac{\mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathrm{LE}}} + \int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\, \frac{\mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathrm{LE}}} - \frac{\mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathrm{LE}}} \int_0^1 \mathrm{d}z\, \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\mathrm{LE}} . \end{aligned}$$

Thus

$$\widehat{\rho} \simeq \widehat{\rho}_{\mathrm{LE}} + \int_0^1 \mathrm{d}z\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\, \widehat{\rho}_{\mathrm{LE}} - \widehat{\rho}_{\mathrm{LE}} \int_0^1 \mathrm{d}z\, \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\mathrm{LE}} . \tag{34}$$

With this approximate density operator, the non-equilibrium expectation value of an observable $\widehat{O}(x)$ becomes

$$\langle\widehat{O}(x)\rangle \simeq \langle\widehat{O}(x)\rangle_{\rm LE} + \int_0^1 {\rm d}z\,\left\langle \widehat{O}(x)\,{\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm LE} - \langle\widehat{O}(x)\rangle_{\rm LE}\int_0^1 {\rm d}z\,\left\langle {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm LE}\,.$$

For two observables $\widehat{O}_1$ and $\widehat{O}_2$, we define the connected local-equilibrium expectation value by

$$\langle\widehat{O}_1\,,\widehat{O}_2\rangle_{c,{\rm LE}} \equiv \langle\widehat{O}_1\widehat{O}_2\rangle_{\rm LE} - \langle\widehat{O}_1\rangle_{\rm LE}\langle\widehat{O}_2\rangle_{\rm LE}\,. \tag{35}$$

Therefore the expectation value can be written compactly as

$$\langle\widehat{O}(x)\rangle \simeq \langle\widehat{O}(x)\rangle_{\rm LE} + \int_0^1 {\rm d}z\,\left\langle \widehat{O}(x)\,,{\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{c,{\rm LE}}\,. \tag{36}$$

This is already an important simplification: a non-equilibrium expectation value has been reduced, at linear order in gradients, to a local-equilibrium expectation value plus a connected correlator computed in local equilibrium.

We can simplify further. Since gradients are the quantities that drive the system away from global equilibrium, the expectation value of a local observable at x should be dominated by the homogeneous equilibrium state defined by the local values $\beta(x)$ and $\zeta(x)$. The dependence on the variation of the thermodynamic fields around x can then be treated perturbatively.

We therefore write

$$\begin{aligned}
\Delta\beta_\nu(y,x) = \beta_\nu(y) - \beta_\nu(x) &= \sum_{n=1}^{\infty}\frac{1}{n!}\left[\prod_{j=1}^{n}\left(y^{\lambda_j}-x^{\lambda_j}\right)\partial^y_{\lambda_j}\right]\beta_\nu(y)\Big|_{y=x}\,,\\
\Delta\zeta(y,x) = \zeta(y) - \zeta(x) &= \sum_{n=1}^{\infty}\frac{1}{n!}\left[\prod_{j=1}^{n}\left(y^{\lambda_j}-x^{\lambda_j}\right)\partial^y_{\lambda_j}\right]\zeta(y)\Big|_{y=x}\,.
\end{aligned} \tag{37}$$

In this way $\beta_\nu(y) = \beta_\nu(x) + \Delta\beta_\nu(y,x)$ and $\zeta(y) = \zeta(x) + \Delta\zeta(y,x)$, so that

$$\begin{aligned}
-\int_\Sigma {\rm d}\Sigma_\mu(y)\left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \widehat{j}^\mu(y)\zeta(y)\right) &= -\beta_\nu(x)\int_\Sigma {\rm d}\Sigma_\mu(y)\widehat{T}^{\mu\nu}(y) + \zeta(x)\int_\Sigma {\rm d}\Sigma_\mu(y)\widehat{j}^\mu(y)\\
&\quad - \int_\Sigma {\rm d}\Sigma_\mu(y)\left(\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y,x) - \widehat{j}^\mu(y)\Delta\zeta(y,x)\right)\\
&\equiv -\beta(x)\cdot\widehat{P} + \zeta(x)\widehat{Q} + \Delta\widehat{\mathcal{E}}\,.
\end{aligned}$$

Equivalently,

$$\Delta\widehat{\mathcal{E}} \equiv -\int_\Sigma {\rm d}\Sigma_\mu(y)\left(\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y,x) - \widehat{j}^\mu(y)\Delta\zeta(y,x)\right)\,. \tag{38}$$

The local equilibrium operator can therefore be separated as

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}}\exp\left(\widehat{\mathcal{E}}_{\rm GE} + \Delta\widehat{\mathcal{E}}\right)\,,\qquad Z_{\rm LE} = {\rm Tr}\left[\exp\left(\widehat{\mathcal{E}}_{\rm GE} + \Delta\widehat{\mathcal{E}}\right)\right]\,, \tag{39}$$

where

$$\widehat{\mathcal{E}}_{\rm GE} \equiv -\beta(x)\cdot\widehat{P} + \zeta(x)\widehat{Q} \tag{40}$$

is the homogeneous global-equilibrium operator constructed from the local values of the thermodynamic fields at the same space-time point x where the observable is evaluated. The correction $\Delta\widehat{\mathcal{E}}$ contains the spatial variation of the thermodynamic fields around x and can be treated perturbatively. At first order,

$$\langle\widehat{O}(x)\rangle_{\rm LE} \simeq \langle\widehat{O}(x)\rangle_{\rm GE} + \int_0^1 {\rm d}z\,\left\langle \widehat{O}(x)\,,{\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\Delta\widehat{\mathcal{E}}\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\,. \tag{41}$$

Substituting this expression into the linear-response formula and keeping only terms that are linear in gradients, namely discarding products such as $\widehat{\mathcal{D}}\Delta\widehat{\mathcal{E}}$, we obtain

$$\langle\widehat{O}(x)\rangle \simeq \langle\widehat{O}(x)\rangle_{\rm GE} + \int_0^1 {\rm d}z\,\left\langle \widehat{O}(x)\,,{\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\Delta\widehat{\mathcal{E}}\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}} + \int_0^1 {\rm d}z\,\left\langle \widehat{O}(x)\,,{\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\,. \tag{42}$$

Thus, in linear response theory, the non-equilibrium expectation value is reduced to the computation of connected correlators in a homogeneous global-equilibrium state. The local values $\beta(x)$ and $\zeta(x)$ determine the reference equilibrium state, while the gradients of these fields generate the corrections.

It is useful to separate the correction into two contributions. We define the *local-equilibrium* correction and the *dissipative* correction as

$$\begin{aligned}\Delta_{\rm LE}O(x) &\equiv \int_0^1 {\rm d}z\, \left\langle \widehat{O}(x)\,, {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\Delta\widehat{\mathcal{E}}\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\,,\\ \Delta_{\rm diss}O(x) &\equiv \int_0^1 {\rm d}z\, \left\langle \widehat{O}(x)\,, {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\,.\end{aligned} \tag{43}$$

Therefore

$$\langle\widehat{O}(x)\rangle \simeq \langle\widehat{O}(x)\rangle_{\rm GE} + \Delta O(x)\,, \tag{44}$$

where the total non-equilibrium correction is

$$\Delta O(x) \equiv \Delta_{\rm LE}O(x) + \Delta_{\rm diss}O(x)\,. \tag{45}$$

II. ENTROPY PRODUCTION

The local equilibrium operator, which we remember is not the real quantum state of the system, is obtained by maximizing the entropy with the constraint of that it must reproduce the local thermodynamic state of the system. The entropy is given by:

$$S = -{\rm Tr}(\widehat{\rho}_{\rm LE}\ln\widehat{\rho}_{\rm LE})\,. \tag{46}$$

Note that being the trace of an operator it must be invariant under local Lorentz transformation and thus is a Lorentz scalar. Also the entropy being a state function may only depend on the internal degrees of freedom of the system and thus cannot depend on the state of motion of it. This is important because some extensions of relativistic thermodynamic assume that the entropy is not a scalar but rather is the time component of a four-vector.

The operator maximizing (46) with the local constraints (24) is again:

$$\widehat{\rho}_{\rm LE}(\tau) = \frac{1}{Z_{\rm LE}}\exp\left[-\int_{\Sigma(\tau)}{\rm d}\Sigma_\mu\left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu\zeta\right)\right]\,,\quad Z_{\rm LE} = {\rm Tr}\left(\exp\left[-\int_{\Sigma(\tau)}{\rm d}\Sigma_\mu\left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu\zeta\right)\right]\right)\,. \tag{47}$$

With the above we can then compute the entropy:

$$\begin{aligned}S &= -{\rm Tr}\left\{\widehat{\rho}_{\rm LE}\left[-\ln Z_{\rm LE} - \int_{\Sigma(\tau)}{\rm d}\Sigma_\mu\left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu\zeta\right)\right]\right\}\\ &= \ln Z_{\rm LE}{\rm Tr}(\widehat{\rho}_{\rm LE}) + \int_{\Sigma(\tau)}{\rm d}\Sigma_\mu\left[{\rm Tr}(\widehat{\rho}_{\rm LE}\,\widehat{T}^{\mu\nu})\beta_\nu - {\rm Tr}(\widehat{\rho}_{\rm LE}\,\widehat{j}^\mu)\zeta\right]\,,\end{aligned}$$

then using that $\widehat{\rho}_{\rm LE}$ is normalized we obtain:

$$S = \ln Z_{\rm LE} + \int_{\Sigma(\tau)}{\rm d}\Sigma_\mu\left(\langle\widehat{T}^{\mu\nu}\rangle_{\rm LE}\,\beta_\nu - \langle\widehat{j}^\mu\rangle_{\rm LE}\,\zeta\right)\,. \tag{48}$$

Note that being ${\rm d}\Sigma_\mu = {\rm d}\Sigma\,\hat{n}_\mu$, due to the constraints (24) the above can be equivalently written as:

$$S = \ln Z_{\rm LE} + \int_{\Sigma(\tau)}{\rm d}\Sigma_\mu\left(\langle\widehat{T}^{\mu\nu}\rangle\,\beta_\nu - \langle\widehat{j}^\mu\rangle\,\zeta\right)\,. \tag{49}$$

that is we can replace the local equilibrium expectation values with the actual non-equilibrium one.

In order to express the total entropy in a manifest covariant way we introduce the entropy current s^μ defined as:

$$S = \int_\Sigma {\rm d}\Sigma_\mu\, s^\mu\,. \tag{50}$$

The integral is independent from Σ if $\partial_\mu s^\mu = 0$ which is equivalent to have:

$$S\Big|_{\Sigma(\tau)} = S\Big|_{\Sigma(\tau')} \,, \tag{51}$$

i.e. no entropy is produced. This of course is true at global equilibrium. We will see how the fact that $\partial_\mu s^\mu \neq 0$ is related to the dissipative term in the linear expansion. This is ultimately the reason that allows to interpret such term as the dissipative one.

Now in order to be able to define an entropy current s^μ, according to (48) and (49) it is clear that it must exist a field ϕ^μ such that:

$$\ln Z_{\rm LE} = \int_\Sigma \mathrm{d}\Sigma_\mu \, \phi^\mu \tag{52}$$

where ϕ^μ is a four-vector called *thermodynamic potential current*. With this we have:

$$S = \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\phi^\mu + \langle \widehat{T}^{\mu\nu} \rangle \, \beta_\nu - \langle \widehat{j}^\mu \rangle \, \zeta \right) \,,$$

which leads to identify:

$$s^\mu = \phi^\mu + \langle \widehat{T}^{\mu\nu} \rangle \beta_\nu - \langle \widehat{j}^\mu \rangle \zeta \,. \tag{53}$$

Note that the entropy current is defined up- to a four-vector L^μ orthogonal to the hypersurface, $L \cdot \hat{n} = 0$.

Now how to prove and derive that ϕ^μ indeed does exist? We start considering the following λ-dependent operator:

$$\widehat{\rho}_{\rm LE}(\tau;\lambda) \equiv \frac{1}{Z_{\rm LE}(\lambda)} \exp\left[-\lambda \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) \right] \tag{54}$$

The above operator is the local equilibrium one with the proper inverse temperature $1/T$ rescaled by λ, that is $1/T \mapsto \lambda/T$ Hence for $\lambda = 1$ the operator is simply the local equilibrium one. Now deriving with respect to λ the logarithm of the partition function:

$$\frac{\partial \ln Z_{\rm LE}(\lambda)}{\partial \lambda} = - \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\langle \widehat{T}^{\mu\nu} \rangle_{\rm LE}(\lambda) \beta_\nu - \langle \widehat{j}^\mu \rangle_{\rm LE}(\lambda) \zeta \right)$$

where the local equilibrium expectation values are computed with the rescaled local equilibrium operator. Now integrating both sides in λ between $\lambda = 1$ and λ_0 we get:

$$\ln Z_{\rm LE} - \ln Z_{\rm LE}(\lambda_0) = - \int_{\lambda_0}^1 \mathrm{d}\lambda \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\langle \widehat{T}^{\mu\nu} \rangle_{\rm LE}(\lambda) \beta_\nu - \langle \widehat{j}^\mu \rangle_{\rm LE}(\lambda) \zeta \right) \,.$$

Now assuming that exists a λ_0 such that $Z_{\rm LE}(\lambda_0) = 1$ then, changing the order of the integrals, we obtain:

$$\ln Z_{\rm LE} = \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left[- \int_{\lambda_0}^1 \mathrm{d}\lambda \left(\langle \widehat{T}^{\mu\nu} \rangle_{\rm LE}(\lambda) \beta_\nu - \langle \widehat{j}^\mu \rangle_{\rm LE}(\lambda) \zeta \right) \right] \,,$$

that is:

$$\phi^\mu \equiv - \int_{\lambda_0}^1 \mathrm{d}\lambda \left(\langle \widehat{T}^{\mu\nu} \rangle_{\rm LE}(\lambda) \beta_\nu - \langle \widehat{j}^\mu \rangle_{\rm LE}(\lambda) \zeta \right) \,.$$

Now the request for the existence of a λ_0 such that $Z_{\rm LE}(\lambda_0) = 1$ is physically the request of the existence of a reference temperature where the system admits an only possible populated state. Indeed this often happens if one consider $\lambda_0 \to +\infty$ which is equivalent to say that for $T = 0$ only one state, the vacuum, is populated. In this regard we can write the thermodnynamic potential current as:

$$\phi^\mu = \int_0^{+\infty} \mathrm{d}\lambda \left(\langle \widehat{T}^{\mu\nu} \rangle_{\rm LE}(\lambda) \beta_\nu - \langle \widehat{j}^\mu \rangle_{\rm LE}(\lambda) \zeta \right) \,. \tag{55}$$

As said the entropy is constant only on an global equilibrium state. For a system at local thermodynamic equilibrium one should expect:

$$S\Big|_{\Sigma(\tau)} \neq S\Big|_{\Sigma(\tau')} \implies \partial_\mu s^\mu \neq 0 \,.$$

In particular if one consider $\tau' \geq \tau$ then according to the second principle of thermodynamic the entropy must increase and one has an effective entropy production:

$$\frac{\delta S}{\delta \tau} \geq 0 \implies \partial_\mu s^\mu \geq 0 \,. \tag{56}$$

In order to compute the rate of entropy production one has to compute the rate of variation with time of the entropy S. The variation is due to the change in the integration hypersurface which evolves from $\Sigma(\tau)$ to $\Sigma(\tau + \delta\tau)$. The variation of the integral for variation of the integration dominion reads:

$$\frac{\delta S}{\delta \tau} = \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \, \xi^\mu \partial_\nu s^\nu + \int_{\partial\Sigma} \mathrm{d}\sigma_{\mu\nu} s^\mu \xi^\nu \,, \tag{57}$$

where $\mathrm{d}\sigma_{\mu\nu} = -\mathrm{d}\sigma_{\nu\mu}$ is the Area two-form associated with the boundary of Σ and $\xi^\mu = \partial x^\mu / \partial \tau$ is the generator of the displacement of the hypersurface. Note that for comoving foliations $\xi^\mu = u^\mu$. If one assumes that the rate of entropy to be vanishing at infinite then the second term reduces to zero:

$$\frac{\delta S}{\delta \tau} = \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \xi^\mu \partial_\nu s^\nu \,. \tag{58}$$

From its definition the entropy can also be written as:

$$S = \ln Z_{\mathrm{LE}} + \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\langle \widehat{T}^{\mu\nu} \rangle \beta_\nu - \langle \widehat{j}^\mu \rangle \zeta \right) \,.$$

Taking the variation with respect to $\delta\tau$ and using again (24) to replace the local equilibrium expectation values with the full non-equilibrium ones:

$$\frac{\delta S}{\delta \tau} = \frac{1}{Z_{\mathrm{LE}}} \frac{\delta Z_{\mathrm{LE}}}{\delta \tau} + \int_\Sigma \mathrm{d}\Sigma_\mu \, \xi^\mu \left(\langle \widehat{T}^{\nu\lambda} \rangle \partial_\nu \beta_\lambda - \langle \widehat{j}^\nu \rangle \partial_\nu \zeta \right) \,. \tag{59}$$

For the variation of Z_{LE} we start considering the following formula:

$$\delta \, \mathrm{e}^{-\widehat{A}} = - \int_0^1 \mathrm{d}z \, \mathrm{e}^{-z\widehat{A}} \delta \widehat{A} \, \mathrm{e}^{-(1-z)\widehat{A}} \,,$$

then:

$$\begin{aligned}
\delta Z_{\mathrm{LE}} &= \mathrm{Tr} \left[\exp \left(- \int_{\Sigma(\tau+\delta\tau)} \mathrm{d}\Sigma \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) \right) \right] - \mathrm{Tr} \left[\exp \left(- \int_{\Sigma(\tau} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) \right) \right] \\
&= \mathrm{Tr} \left\{ \exp \left(- \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) \right) \left[- \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \, \xi^\mu \left(\widehat{T}^{\nu\lambda} \partial_\nu \beta_\lambda - \widehat{j}^\nu \partial_\nu \zeta \right) \right] \right\} \delta\tau + \mathcal{O}(\delta\tau^2) \,.
\end{aligned}$$

Dividing by Z_{LE} and retaining only the $\delta\tau$ term we thus get:

$$\begin{aligned}
\frac{1}{Z_{\mathrm{LE}}} \frac{\delta Z_{\mathrm{LE}}}{\delta \tau} &= - \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \xi^\mu \, \mathrm{Tr} \left[\widehat{\rho}_{\mathrm{LE}} \left(\widehat{T}^{\nu\lambda} \partial_\nu \beta_\lambda - \widehat{j}^\nu \partial_\nu \zeta \right) \right] \\
&= - \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \, \xi^\mu \left(\langle \widehat{T}^{\nu\lambda} \rangle_{\mathrm{LE}} \partial_\nu \beta_\lambda - \langle \widehat{j}^\nu \rangle_{\mathrm{LE}} \partial_\nu \zeta \right) \,.
\end{aligned}$$

Then plugging the above in (59) we obtain:

$$\frac{\delta S}{\delta \tau} = - \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \, \xi^\mu \left(\langle \widehat{T}^{\nu\lambda} \rangle_{\mathrm{LE}} \partial_\nu \beta_\lambda - \langle \widehat{j}^\nu \rangle_{\mathrm{LE}} \partial_\nu \zeta \right) + \int_\Sigma \mathrm{d}\Sigma_\mu \, \xi^\mu \left(\langle \widehat{T}^{\nu\lambda} \rangle \partial_\nu \beta_\lambda - \langle \widehat{j}^\nu \rangle \partial_\nu \zeta \right) \,. \tag{60}$$

Combining the above with (58) we thus finally arrive at:

$$\partial_\mu s^\mu = \left(\langle \widehat{T}^{\lambda\nu}\rangle - \langle \widehat{T}^{\lambda\nu}\rangle_{\rm LE}\right)\partial_\lambda\beta_\nu - \left(\langle \widehat{j}^{\nu}\rangle - \langle \widehat{j}^{\nu}\rangle_{\rm LE}\right)\partial_\nu\zeta \ . \tag{61}$$

Thus the entropy production is driven by the difference between the full non-equilibrium and the local equilibrium expectation values. If I use linear response theory then:

$$\begin{aligned}\langle \widehat{T}^{\mu\nu}\rangle &\simeq \langle \widehat{T}^{\mu\nu}\rangle_{\rm GE} + \Delta_{\rm LE}T^{\mu\nu} + \Delta_{\rm diss}T^{\mu\nu} \ , \\ \langle \widehat{j}^{\mu}\rangle &\simeq \langle \widehat{j}^{\mu}\rangle_{\rm GE} + \Delta_{\rm LE}j^{\mu} + \Delta_{\rm diss}j^{\mu} \ ,\end{aligned}$$

and thus:

$$\partial_\mu s^\mu = \Delta_{\rm diss}\, T^{\lambda\nu}\partial_\lambda\beta_\nu - \Delta_{\rm diss}\, j^{\nu}\partial_\nu\zeta \ . \tag{62}$$

It is clear then the terms we interpreted as dissipative are indeed those that drive the entropy rate production.

III. KUBO FORMULAE AND DISSIPATIVE CORRECTIONS

Now the dissipative and local equilibrium corrections can be both computed in linear response theory according to the formulae (43). However, despite being structuraly similar, they are physically different; as we showed the dissipative part produces entropy. For a general observable $\widehat{O}$ this is defined as:

$$\Delta_{\rm diss}O(x) = \int_0^1 {\rm d}z \int_\Omega {\rm d}\Omega(y)\,\partial_\mu\beta_\nu(y)\left\langle \widehat{O}(x)\,{\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{T}^{\mu\nu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm GE} - \left\langle \widehat{O}(x)\right\rangle_{\rm GE}\left\langle \widehat{T}^{\mu\nu}(x)\right\rangle_{\rm GE} \ . \tag{63}$$

The above in the end reduces to the connected part of the expectation value which is defined as:

$$\left\langle \widehat{O}_1(x)\,,\widehat{O}_2(x)\right\rangle_{c,{\rm GE}} = \left\langle \widehat{O}_1(x)\,\widehat{O}_2(x)\right\rangle_{\rm GE} - \left\langle \widehat{O}_1(x)\right\rangle_{\rm GE}\left\langle \widehat{O}_2(x)\right\rangle_{\rm GE} \ . \tag{64}$$

Now lets concentrate on the thermal expectation value in the first term. Under space-time translation the stress-energy tensor transforms according to:

$${\rm e}^{{\rm i}y\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(x)\,{\rm e}^{-{\rm i}y\cdot\widehat{P}} = \widehat{T}^{\mu\nu}\,(x+y) \ , \tag{65}$$

then:

$${\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{T}^{\mu\nu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}} = {\rm e}^{{\rm i}({\rm i}z\beta)\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(y)\,e^{-{\rm i}({\rm i}z\beta)\cdot\widehat{P}} = \widehat{T}^{\mu\nu}\,(y+{\rm i}z\beta(x)) \ .$$

In order to perform the integration we approximate the region Ω with the region of space-time Ω' between the flat hyperplane passing through x and parallel to $\Sigma_{\rm D}$ and the hyperplane passing through x_0 which is the translation of x on the equilibrium hypersurface Σ_0 (see figure). Following this geometrical approximation one can express the measure of the four-volume integral using cartesian coordinates and $\int {\rm d}\Omega = \int {\rm d}^4y$. If one denotes the direction of the plane with $\boldsymbol{\xi}$ and the orthogonal direction with τ, then one can parametrize the vector y^μ as follows:

$$y^\mu = (\tau,\boldsymbol{\xi}) \ , \quad {\rm d}^4y = {\rm d}\tau\,{\rm d}^3\boldsymbol{\xi} \ . \tag{66}$$

Being $\boldsymbol{\xi}$ the space-coordinate of an hyperplane it is invariant under translation and one has that in this coordinates:

$$\begin{aligned}\widehat{T}^{\mu\nu}\,(y+{\rm i}z\beta(x)) &= \widehat{T}^{\mu\nu}\left(\tau+{\rm i}z\sqrt{\beta^2(x)},\boldsymbol{\xi}\right) = \widehat{T}^{\mu\nu}\left(\tau_0+{\rm i}z\sqrt{\beta^2(x)},\boldsymbol{\xi}\right) + \int_{\tau_0}^{\tau}{\rm d}\vartheta\frac{\partial}{\partial\vartheta}\widehat{T}^{\mu\nu}\left(\vartheta+{\rm i}z\sqrt{\beta^2(x)},\boldsymbol{\xi}\right) \\ &= \widehat{T}^{\mu\nu}\left(\tau_0+{\rm i}z\sqrt{\beta^2(x)},\boldsymbol{\xi}\right) + \int_{\tau_0}^{\tau}{\rm d}z\frac{1}{{\rm i}\sqrt{\beta^2(x)}}\frac{\partial}{\partial z}\widehat{T}^{\mu\nu}\left(\vartheta+{\rm i}z\sqrt{\beta^2(x)},\boldsymbol{\xi}\right) \ .\end{aligned}$$

Plugging the above in (63) then gives:

$$\begin{aligned}&\int_0^1{\rm d}z\left\langle \widehat{O}(x)\,{\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{T}^{\mu\nu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm GE} = \int_0^1{\rm d}z\left\langle \widehat{O}(x)\widehat{T}^{\mu\nu}\left(\tau_0+{\rm i}z\sqrt{\beta^2(x)},\boldsymbol{\xi}\right)\right\rangle_{\rm GE} \\ &+\int_{\tau_0}^{\tau}{\rm d}\vartheta\frac{1}{{\rm i}\sqrt{\beta^2(x)}}\left[\left\langle \widehat{O}(x)\widehat{T}^{\mu\nu}\left(\vartheta+{\rm i}\sqrt{\beta^2(x)},\boldsymbol{\xi}\right)\right\rangle_{\rm GE} - \left\langle \widehat{O}(x)\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\right\rangle_{\rm GE}\right] \ ,\end{aligned}$$

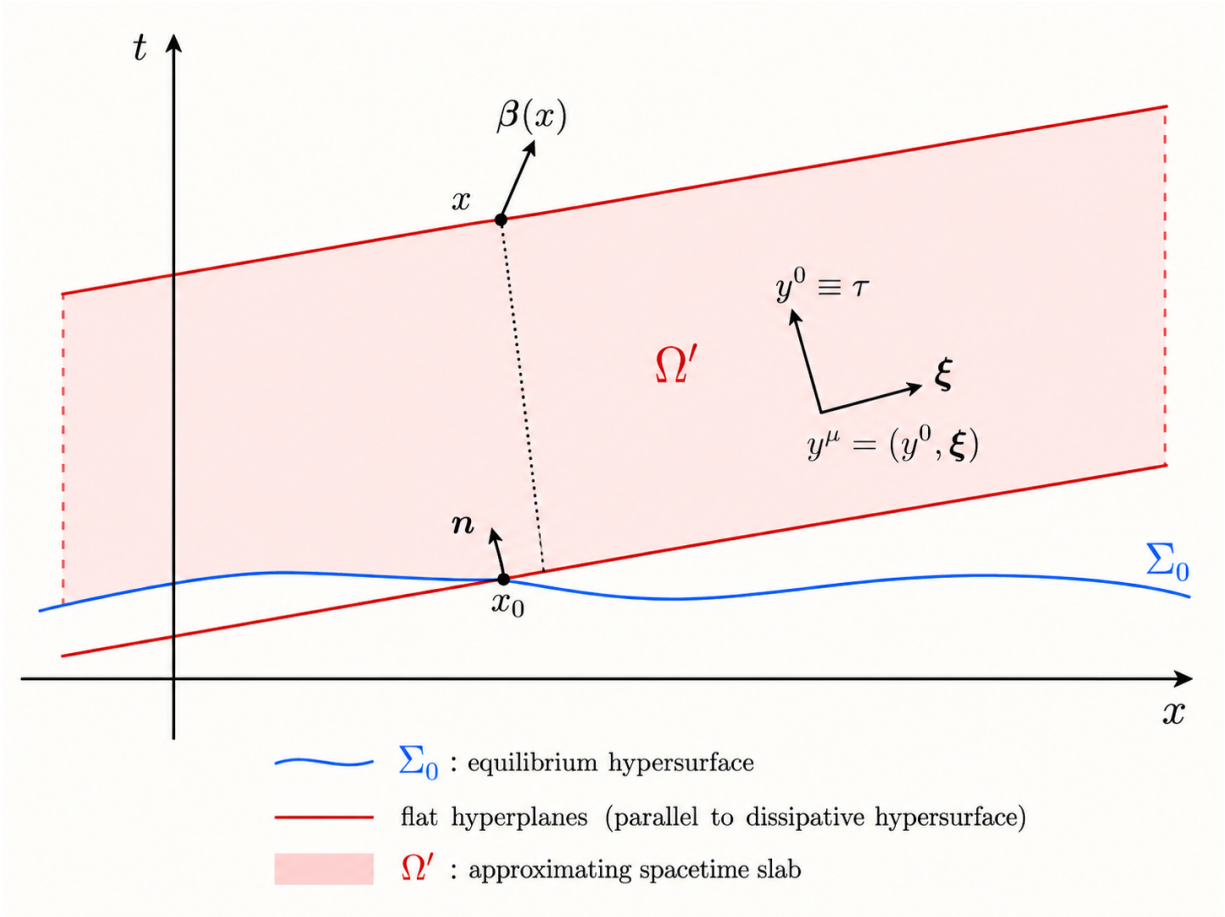


FIG. 1: Geometrical approximation used in the derivation of the Kubo formulae. The original integration region Ω is approximated by the spacetime slab Ω' bounded by two flat hyperplanes parallel to the dissipative hypersurface. The lower hyperplane passes through the point $x_0 \in \Sigma_0$, while the upper one passes through the spacetime point x. The coordinates $y^\mu = (y^0, \boldsymbol{\xi})$ are adapted to these flat hyperplanes: $y^0 \equiv \tau$ labels the direction orthogonal to the hyperplanes, whereas $\boldsymbol{\xi}$ denotes the spatial coordinates intrinsic to each hyperplane.

where in the second term we integrate over $\mathrm{d}z$. Now the above further simplifies. Consider first:

$$\widehat{T}^{\mu\nu}\left(\vartheta + \mathrm{i}\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right) = \mathrm{e}^{\mathrm{i}(\mathrm{i}\beta)\cdot\widehat{P}}\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\,\mathrm{e}^{-\mathrm{i}(\mathrm{i}\beta)\cdot\widehat{P}} = \mathrm{e}^{-\beta\cdot\widehat{P}}\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\,\mathrm{e}^{\beta\cdot\widehat{P}}\,,$$

then:

$$\begin{aligned}
\left\langle \widehat{O}(x)\widehat{T}^{\mu\nu}\left(\vartheta + \mathrm{i}\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right)\right\rangle_{\mathrm{GE}} &= \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}\,\widehat{O}(x)\,\mathrm{e}^{-\beta\cdot\widehat{P}}\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\,\mathrm{e}^{\beta\cdot\widehat{P}}\right] \\
&= \frac{1}{Z_{\mathrm{GE}}}\mathrm{Tr}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{O}(x)\,\mathrm{e}^{-\beta\cdot\widehat{P}}\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\,\mathrm{e}^{\beta\cdot\widehat{P}}\right] \\
&= \frac{1}{Z_{\mathrm{GE}}}\mathrm{Tr}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\widehat{O}(x)\right] \\
&= \left\langle \widehat{T}^{\mu\nu}\,(\vartheta,\boldsymbol{\xi})\;\widehat{O}(x)\right\rangle_{\mathrm{GE}}\,,
\end{aligned}$$

so that:

$$\left\langle \widehat{O}(x)\widehat{T}^{\mu\nu}\left(\vartheta + \mathrm{i}\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right)\right\rangle_{\mathrm{GE}} - \left\langle \widehat{O}(x)\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi})\right\rangle_{\mathrm{GE}} = \left\langle\left[\widehat{T}^{\mu\nu}(\vartheta,\boldsymbol{\xi}), \widehat{O}(x)\right]\right\rangle_{\mathrm{GE}}\,. \tag{67}$$

From the above we thus obtain that the first piece in the dissipative correction is:

$$\begin{aligned}
&\int_0^1 \mathrm{d}z \int_\Omega \mathrm{d}\Omega(y)\, \partial_\mu \beta_\nu(y) \left\langle \widehat{O}(x)\, \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{T}^{\mu\nu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{\mathrm{GE}} \\
&= \int \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_0^1 \mathrm{d}z\, \partial_\mu \beta_\nu(y) \left\langle \widehat{O}(x) \widehat{T}^{\mu\nu}\left(\tau_0 + \mathrm{i}z\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right) \right\rangle_{\mathrm{GE}} \\
&+ \mathrm{i}T(x) \int \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta\, \partial_\mu \beta_\nu(y) \left\langle \left[\widehat{T}^{\mu\nu}(\vartheta, \boldsymbol{\xi}), \widehat{O}(x) \right] \right\rangle_{\mathrm{GE}} ,
\end{aligned}$$

which finally implies:

$$\begin{aligned}
\Delta_{\mathrm{diss}} O(x) = &\int \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_0^1 \mathrm{d}z\, \partial_\mu \beta_\nu(y) \left\langle \widehat{O}(x) \widehat{T}^{\mu\nu}\left(\tau_0 + \mathrm{i}z\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right) \right\rangle_{\mathrm{GE}} \\
&+ \mathrm{i}T(x) \int \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta\, \partial_\mu \beta_\nu(y) \left\langle \left[\widehat{T}^{\mu\nu}(\vartheta, \boldsymbol{\xi}), \widehat{O}(x) \right] \right\rangle_{\mathrm{GE}} - \left\langle \widehat{O}(x) \right\rangle_{\mathrm{GE}} \left\langle \widehat{T}^{\mu\nu}(x) \right\rangle_{\mathrm{GE}} .
\end{aligned} \tag{68}$$

As we saw the above is directly related with the connected part of the expectation value between the observable and the stress-energy tensor. Now one fundamental assumption in deriving Kubo formulae is that we usually deals with local operators $\widehat{O}_1(x_1)$ and $\widehat{O}_2(x_2)$. The locality means that the correlation between the two is expected to be vanishing over large enough distances. Namely if the typical length scale of the system is λ_β then for $|x_1 - x_2| \gtrsim \lambda_\beta$ any kind of microscopic correlation is bounded to be vanishing. From a statistical point of view this implies:

$$\left\langle \widehat{O}_1(x_1) \widehat{O}_2(x_2) \right\rangle_{\mathrm{GE}} - \left\langle \widehat{O}_1(x_1) \right\rangle_{\mathrm{GE}} \left\langle \widehat{O}_2(x_2) \right\rangle_{\mathrm{GE}} \to 0 , \quad \text{for } |x_1 - x_2| \gtrsim \lambda_\beta . \tag{69}$$

Hence:

$$\begin{aligned}
\left\langle \widehat{O}(x) \widehat{T}^{\mu\nu}\left(\tau_0 + \mathrm{i}z\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right) \right\rangle_{\mathrm{GE}} &\simeq \left\langle \widehat{O}(x) \right\rangle_{\mathrm{GE}} \left\langle \widehat{T}^{\mu\nu}\left(\tau_0 + \mathrm{i}z\sqrt{\beta^2(x)}, \boldsymbol{\xi}\right) \right\rangle_{\mathrm{GE}} \\
&= \left\langle \widehat{O}(x) \right\rangle_{\mathrm{GE}} \left\langle \widehat{T}^{\mu\nu}(x) \right\rangle_{\mathrm{GE}} ,
\end{aligned}$$

where in the last equality we used again that the global equilibrium state is homogeneous. Plugging the above in (68) we obtain that the disconnected term cancels out and we are simply left out with:

$$\Delta_{\mathrm{diss}} O(x) = -\mathrm{i}T(x) \int \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta\, \partial_\mu \beta_\nu(y) \left\langle \left[\widehat{T}^{\mu\nu}(\vartheta, \boldsymbol{\xi}), \widehat{O}(x) \right] \right\rangle_{\mathrm{GE}} . \tag{70}$$

Now the assumption of local operators has another important consequence. Indeed the dissipative correction (70) is a convolution over the plasma phase of a microscopic commutator and the gradient of a thermodynamic field. The two quantities are generally expected to vary of much different scales. The first one will vary over microscopic scales whereas the second one over scales λ_β which, in hydrodynamic regime, are assumed to be much bigger. This means that, transforming the thermodynamic field in Fourier space:

$$\partial_\mu \beta_\nu(y) = \int \mathrm{d}^4 k\, \mathrm{e}^{-\mathrm{i}k\cdot y} \mathcal{B}_{\mu\nu}(k) , \tag{71}$$

the Fourier mode $\mathcal{B}$ is expected to be an extremely peeked function of k for $k = 0$. Hence plugging in it in(70) we get:

$$\Delta_{\mathrm{diss}} O(x) = -\mathrm{i}T(x) \int \mathrm{d}^4 k\, \mathrm{e}^{-\mathrm{i}k\cdot x} \mathcal{B}_{\mu\nu}(k) \left\{ \int \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta \left\langle \left[\widehat{T}^{\mu\nu}(\vartheta, \boldsymbol{\xi}), \widehat{O}(x) \right] \mathrm{e}^{-\mathrm{i}k\cdot(y-x)} \right\rangle_{\mathrm{GE}} \right\},$$

where the term in the curvy brackets is expected to be a smooth function of k so that the convolution in the Fourier space is dominated by its Taylor expansion around $k = 0$.

Consider first

$$\begin{aligned}
&\int_{\tau_0}^{\tau_x} \mathrm{d}\tau \mathrm{d}^3\boldsymbol{\xi} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta \left\langle \left[\widehat{T}^{\mu\nu}(\vartheta, \boldsymbol{\xi}), \widehat{O}(x) \right] \mathrm{e}^{-\mathrm{i}k\cdot(y-x)} \right\rangle_{\mathrm{GE}} \\
&= \int_{\tau_0}^{\tau_x} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k^0(\tau - \tau_x)} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta \left\{ \int \mathrm{d}^3\boldsymbol{\xi} \left\langle \left[\widehat{O}(x), \widehat{T}^{\mu\nu}(\vartheta, \boldsymbol{\xi}) \right] \right\rangle_{\mathrm{GE}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\boldsymbol{\xi} - \mathbf{x})} \right\} \\
&\equiv \int_{\tau_0}^{\tau_x} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k^0(\tau - \tau_x)} \int_{\tau_0}^{\tau} \mathrm{d}\vartheta\, F(\vartheta) \equiv \int_{\tau_0}^{\tau_x} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k^0(\tau - \tau_x)}\, G(\tau) .
\end{aligned}$$

This last term can be work out as follows:

$$\begin{aligned}\int_{\tau_0}^{\tau_x} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)}\, G(\tau) &= \int_{\tau_0}^{\tau_x} \mathrm{d}\tau G(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)}}{-\mathrm{i}k^0}\right) \\ &= \frac{\mathrm{i}}{k^0}\int_{\tau_0}^{\tau_x} \mathrm{d}\tau \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(G(\tau)\mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)}\right) - \mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)}\frac{\mathrm{d}\, G(\tau)}{\mathrm{d}\tau}\right] \\ &= \frac{\mathrm{i}}{k^0}\left[\int_{\tau_0}^{\tau_x} \mathrm{d}\vartheta\, F(\vartheta) - \int_{\tau_0}^{\tau_x} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)} F(\tau)\right] ,\end{aligned}$$

where we used that $G(\tau)$ is the integral function of F from τ_0 to τ and thus $G(\tau_0) = 0$. It then follows:

$$\Delta_{\mathrm{diss}} O(x) = \mathrm{i}T(x)\int \mathrm{d}^4k\, \mathrm{e}^{-\mathrm{i}k\cdot x}\mathcal{B}_{\mu\nu}(k)\left\{\frac{\mathrm{i}}{k^0}\int_{\tau_0}^{\tau_x}\mathrm{d}\tau\mathrm{d}^3\boldsymbol{\xi}\int_{\tau_0}^{\tau}\mathrm{d}\vartheta\, \left\langle\left[\widehat{O}(x)\,,\widehat{T}^{\mu\nu}(\tau,\boldsymbol{\xi})\right]\right\rangle_{\mathrm{GE}} \mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{x})}\right\} . \tag{72}$$

As we pointed out the Fourier transform of the gradient of the thermodynamic field, being extremely peeked for $k = 0$, selects for the term in curly brackets, the contribution coming from $k \to 0$. We can then approximate the term with:

$$\Delta_{\mathrm{diss}} O(x) = T(x)\partial_\mu\beta_\nu(x)\frac{\mathrm{d}}{\mathrm{d}k^0}\lim_{\mathbf{k}\to 0}\left\{\int_{\tau_0}^{\tau_x}\mathrm{d}\tau\mathrm{d}^3\boldsymbol{\xi}\int_{\tau_0}^{\tau}\mathrm{d}\vartheta\, \left\langle\left[\widehat{O}(x)\,,\widehat{T}^{\mu\nu}(\tau,\boldsymbol{\xi})\right]\right\rangle_{\mathrm{GE}} \mathrm{e}^{-\mathrm{i}k^0(\tau-\tau_x)}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{x})}\right\}_{k^0=0} .$$

A. Transport Coefficients

Lets start considering the general dissipative correction:

$$\begin{aligned}\Delta_{\mathrm{diss}} O(x) = {} & T(x)\partial_\mu\beta_\nu(x)\frac{\mathrm{d}}{\mathrm{d}k^0}\lim_{\mathbf{k}\to 0}\int_{-\infty}^{0}\mathrm{d}^4z\left\langle\left[\widehat{O}(0),\widehat{T}^{\mu\nu}(z)\right]\right\rangle_{\mathrm{GE}}\mathrm{e}^{-\mathrm{i}k\cdot z}\Big|_{k^0=0} \\ & - T(x)\partial_\mu\zeta(x)\frac{\mathrm{d}}{\mathrm{d}k^0}\lim_{\mathbf{k}\to 0}\int_{-\infty}^{0}\mathrm{d}^4z\left\langle\left[\widehat{O}(0),\widehat{j}^{\mu}(z)\right]\right\rangle_{\mathrm{GE}}\mathrm{e}^{-\mathrm{i}k\cdot z}\Big|_{k^0=0} .\end{aligned} \tag{73}$$

The microscopic correlators contain information about the microscopic interaction and are directly related to the so called *dissipative coefficients*

add here discussion about transport eq. and dissipative coefficients

The number of these scalar coefficients strongly depends on the geometry and nature of the observable we are considering.

We start considering the case of a scalar observable. From covariance it must follows that:

$$\Delta_{\mathrm{diss}} O(x) = X^{\mu\nu}\partial_\mu\beta_\nu(x) + Y^\mu\partial_\mu\zeta(x) . \tag{74}$$

The two tensors X and Y depends in principle on the temperature, chemical potential and other thermodynamic fields but they do not depends on the coordinates. They can be decomposed taking into account that they originate from the equilibrium expectation value of the observable itslef with the conserved densities:

$$\begin{aligned}X^{\mu\nu} &\equiv T(x)\frac{\mathrm{d}}{\mathrm{d}k^0}\lim_{\mathbf{k}\to 0}\int_{-\infty}^{0}\mathrm{d}^4z\left\langle\left[\widehat{O}(0),\widehat{T}^{\mu\nu}(z)\right]\right\rangle_{\mathrm{GE}}\mathrm{e}^{-\mathrm{i}k\cdot z}\Big|_{k^0=0} , \\ Y^{\mu} &\equiv T(x)\frac{\mathrm{d}}{\mathrm{d}k^0}\lim_{\mathbf{k}\to 0}\int_{-\infty}^{0}\mathrm{d}^4z\left\langle\left[\widehat{O}(0),\widehat{j}^{\mu}(z)\right]\right\rangle_{\mathrm{GE}}\mathrm{e}^{-\mathrm{i}k\cdot z}\Big|_{k^0=0} ,\end{aligned} \tag{75}$$

hence they can be decomposed on the most general tensor (vector) we can build with β^μ, $g^{\mu\nu}$ and eventually $\epsilon^{\mu\nu\rho\sigma}$. One can work with two equivalent bases:

$$\begin{aligned}&\beta^\mu\,, \quad g^{\mu\nu}\,, \quad \epsilon^{\mu\nu\rho\sigma}\,, \\ &u^\mu\,, \quad \Delta^{\mu\nu}\,, \quad \epsilon^{\mu\nu\rho\sigma}\,,\end{aligned} \tag{76}$$

where $u^\mu = \beta^\mu/\sqrt{\beta^2}$ is the four-velocity while:

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu \ , \tag{77}$$

is the transverse tensor.The two bases are equivalent, but they are useful for different purposes. The basis built with β^μ and $g^{\mu\nu}$ is closer to the statistical operator, since β^μ is the Lagrange multiplier coupled to the stress-energy tensor. Therefore it is natural when one derives the Kubo formulae from the microscopic theory.

On the other hand, the basis built with u^μ and $\Delta^{\mu\nu}$ is more convenient for the hydrodynamic interpretation. Indeed, it separates the longitudinal direction, selected by the local rest frame of the fluid, from the transverse spatial directions. In this basis the tensor structures are immediately organized into scalar, vector and traceless tensor sectors under spatial rotations in the local rest frame. This is the same organization used in dissipative hydrodynamics. Therefore, after the microscopic coefficients have been obtained, the $u^\mu, \Delta^{\mu\nu}$ basis is the natural one for identifying the shear viscosity, bulk viscosity and conductivity coefficients.

Given that the stress-energy tensor is symmetric then the only possible expansion is:

$$\Delta_{\rm diss} O(x) = X_1 u^\mu(x) u^\nu(x) \partial_\mu \beta_\nu(x) + X_2 \Delta^{\mu\nu}(x) \partial_\mu \beta_\nu(x) + Y_1 u^\mu(x) \partial_\mu \zeta(x) \ . \tag{78}$$

One defines the longitudinal, or flow, and transverse derivative as follow (see Lessons by Shokri-Chiarini):

$$D \equiv u^\mu \partial_\mu \ , \quad \partial_\perp^\mu \equiv \Delta^{\mu\nu} \partial_\nu \ , \tag{79}$$

and we denote with a dot the flow derivative of a scalar given that in the rest frame of the fluid D reduces to the usual time derivative. The two derivatives satisfy the following relations:

$$u^\mu D(u_\mu) = 0 \ , \quad u_\mu \partial_\perp^\mu v = 0 \ \forall \, v \ .$$

Also we define the expansion scalar as:

$$\theta \equiv \partial \cdot u \ . \tag{80}$$

Then, computing the longitudinal and transverse derivatives of the four-temperature we obtain:

$$\Delta_{\rm diss} O(x) = -X_1 \frac{\dot T}{T^2} + Y_1 \, \dot\zeta + X_2 \frac{\theta}{T} \ . \tag{81}$$

So in principle, for a scalar observable, the dissipative correction depends on three unknown coefficients. In the following we will prove that these three coefficients are indeed related one with the other once we consider the equation of motion of hydrodynamics. Better to say once one take into accounts the conservation equation and eventually the equation of state then the coefficients are said to be *on shell* and turns out to be not all independent.

To see this let us use the leading order expressions for the four-current and stress-energy tensor, namely their ideal form:

$$T^{\mu\nu}_{\rm id} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu} \ , \qquad j^\mu_{\rm id} = n u^\mu \ . \tag{82}$$

The corresponding equations of motion are obtained from $\partial_\mu T^{\mu\nu}_{\rm id} = 0$ and $\partial_\mu j^\mu_{\rm id} = 0$. Projecting the conservation of the stress-energy tensor along u^μ, and using charge conservation, one obtains

$$\dot\varepsilon + (\varepsilon + p)\theta = 0 \ , \qquad \dot n + n\theta = 0 \ . \tag{83}$$

Now, since the energy density, pressure and particle density depend on the intensive thermodynamic parameters T and ζ, we have

$$\dot\varepsilon = \dot T \, \partial_T \varepsilon + \dot\zeta \, \partial_\zeta \varepsilon \ , \qquad \dot n = \dot T \, \partial_T n + \dot\zeta \, \partial_\zeta n \ .$$

Then the equations of motion (83) can be cast in the following matrix form:

$$\begin{pmatrix} \partial_T \varepsilon & \partial_\zeta \varepsilon \\ \partial_T n & \partial_\zeta n \end{pmatrix} \begin{pmatrix} \dot T \\ \dot\zeta \end{pmatrix} = -\theta \begin{pmatrix} \varepsilon + p \\ n \end{pmatrix} \ . \tag{84}$$

Defining

$$J \equiv \partial_T \varepsilon \, \partial_\zeta n - \partial_\zeta \varepsilon \, \partial_T n \ , \tag{85}$$

the above system can be solved in favor of $\dot{T}$ and $\dot{\zeta}$:

$$\dot{T} \equiv -\mathcal{A}\theta = -\frac{(\varepsilon+p)\partial_\zeta n - n\partial_\zeta \varepsilon}{\partial_T \varepsilon\, \partial_\zeta n - \partial_\zeta \varepsilon\, \partial_T n}\theta \,, \tag{86}$$

and

$$\dot{\zeta} \equiv -\mathcal{B}\theta = -\frac{n\partial_T \varepsilon - (\varepsilon+p)\partial_T n}{\partial_T \varepsilon\, \partial_\zeta n - \partial_\zeta \varepsilon\, \partial_T n}\theta \,. \tag{87}$$

Plugging these relations into the dissipative correction to the scalar operator gives

$$\Delta_{\rm diss} O(x) = \chi_O \theta \,, \qquad \chi_O \equiv \frac{X_1 \mathcal{A}}{T^2} + \frac{X_2}{T} - Y_1 \mathcal{B} \,. \tag{88}$$

So overall, the scalar example shows in the simplest possible way the meaning of the on-shell reduction. Before using the hydrodynamic equations, the dissipative correction contains three different scalar gradients, namely $\dot{T}$, $\dot{\zeta}$ and θ. These gradients are independent from the point of view of symmetry alone. However, once the ideal equations of motion are imposed, $\dot{T}$ and $\dot{\zeta}$ are fixed in terms of the expansion scalar θ. Therefore the whole scalar correction reduces to a single scalar structure,

$$\Delta_{\rm diss} O(x) = \chi_O \, \theta \,. \tag{89}$$

The coefficient χ_O is the on-shell dissipative coefficient associated with the scalar operator O. It is not equal to one of the microscopic coefficients X_1, X_2 or Y_1 separately, but to the particular combination selected by the conservation laws,

$$\chi_O = \frac{X_1 \mathcal{A}}{T^2} + \frac{X_2}{T} - Y_1 \mathcal{B} \,. \tag{90}$$

This is the general lesson that will be used below for the stress-energy tensor and for the current: the tensor decomposition gives the most general off-shell structures allowed by symmetry, while the usual hydrodynamic transport coefficients are obtained only after using the ideal equations of motion.

We are particularly interested in considering the dissipative corrections to the conserved densities themselves, namely the stress-energy tensor and the four-current. From the general expression for $\Delta_{\rm diss} O$ one can clearly see that $\Delta_{\rm diss} T^{\mu\nu}$ and $\Delta_{\rm diss} j^\mu$ are directly related to entropy production.

Let us start from the former. We have

$$\Delta_{\rm diss} T^{\mu\nu}(x) = X^{\mu\nu\rho\sigma} \partial_\rho \beta_\sigma(x) - Y^{\mu\nu\sigma} \partial_\sigma \zeta(x) \,. \tag{91}$$

The two tensor coefficients X and Y are given by

$$\begin{aligned} X^{\mu\nu,\,\rho\sigma} &\equiv T(x) \frac{\mathrm{d}}{\mathrm{d}k^0} \lim_{\mathbf{k}\to 0} \int_{-\infty}^{0} \mathrm{d}^4 z \, \left\langle \left[\widehat{T}^{\mu\nu}(0), \widehat{T}^{\rho\sigma}(z) \right] \right\rangle_{\rm GE} \mathrm{e}^{-\mathrm{i}k\cdot z} \Big|_{k^0=0} \,, \\ Y^{\mu\nu,\,\sigma} &\equiv T(x) \frac{\mathrm{d}}{\mathrm{d}k^0} \lim_{\mathbf{k}\to 0} \int_{-\infty}^{0} \mathrm{d}^4 z \, \left\langle \left[\widehat{T}^{\mu\nu}(0), \widehat{j}^{\sigma}(z) \right] \right\rangle_{\rm GE} \mathrm{e}^{-\mathrm{i}k\cdot z} \Big|_{k^0=0} \,. \end{aligned} \tag{92}$$

The stress-energy tensor is symmetric, hence the tensor X is symmetric under $\mu \leftrightarrow \nu$ and under $\rho \leftrightarrow \sigma$, whereas Y is symmetric in the first two indices. The goal is to expand these tensors into scalar structures in the same way as we did for the scalar operator. Starting from X, one has in principle $10 \times 10 = 100$ components, 10 for each symmetric pair of indices. However, the independent scalar structures are much fewer.

In the local rest frame, being the equilibrium state homogeneous and isotropic, the tensors X and Y must decompose into rotational invariant objects. A symmetric tensor can be decomposed into irreducible representations of the rotation group as

$$T^{\mu\nu} = 0 \oplus 0 \oplus 1 \oplus 2 \,. \tag{93}$$

The number identifies the spin label. Spin label 0 has dimension one and is a scalar, spin label 1 has dimension three and is a vector, and spin label 2 has dimension five and is a symmetric traceless spatial tensor. The two scalars are

the T^{00} component and the spatial trace. The vector component is the mixed time-space component T^{0i}, whereas the spin-two component is the symmetric traceless tensor

$$T^{ij} - \frac{1}{3}\delta^{ij}T^k_k \ . \tag{94}$$

Thus, schematically,

$$T^{\mu\nu} \mapsto T^{00} \oplus T^i_i \oplus T^{0i} \oplus \left(T^{ij} - \frac{1}{3}\delta^{ij}T^k_k\right) \ , \tag{95}$$

namely $10 = 1 + 1 + 3 + 5$. The four-current has a simpler decomposition, being a four-vector:

$$j^\mu = 0 \oplus 1 \ , \tag{96}$$

where the scalar is j^0 and the vector is j^i. With these expansions one has

$$\begin{aligned} X^{\mu\nu,\,\rho\sigma} &= (0 \oplus 0 \oplus 1 \oplus 2) \otimes (0 \oplus 0 \oplus 1 \oplus 2) \ , \\ Y^{\mu\nu,\,\sigma} &= (0 \oplus 0 \oplus 1 \oplus 2) \otimes (0 \oplus 1) \ . \end{aligned} \tag{97}$$

For X there are therefore six scalar structures: four from the scalar sector $0 \otimes 0$, one from the scalar product $1 \otimes 1$, and one from the double contraction $2 \otimes 2$. For Y there are three scalar structures.

In covariant language, equilibrium selects a preferred time-like direction, namely

$$u^\mu = \frac{\beta^\mu}{\sqrt{\beta^2}} \ . \tag{98}$$

Homogeneity and isotropy require invariance under rotations in the plane orthogonal to u^μ. Therefore the natural basis is built out of the two projectors

$$u^\mu u^\nu \ , \qquad \Delta^{\mu\nu} \ .$$

The most general decomposition of X compatible with the above symmetries is

$$\begin{aligned} X^{\mu\nu,\,\rho\sigma} = {} & X_1\, u^\mu u^\nu u^\rho u^\sigma + X_2\, u^\mu u^\nu \Delta^{\rho\sigma} + X_3\, \Delta^{\mu\nu} u^\rho u^\sigma + X_4\, \Delta^{\mu\nu}\Delta^{\rho\sigma} \\ & + X_5\, (u^\mu u^\rho \Delta^{\nu\sigma} + u^\mu u^\sigma \Delta^{\nu\rho} + u^\nu u^\rho \Delta^{\mu\sigma} + u^\nu u^\sigma \Delta^{\mu\rho}) \\ & + X_6\, (\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho}) \ . \end{aligned} \tag{99}$$

Contracting this expression with $\partial_\rho \beta_\sigma$ gives the following useful identities:

$$\begin{aligned} u^\mu u^\nu u^\rho u^\sigma \partial_\rho \beta_\sigma &= -u^\mu u^\nu \frac{\dot{T}}{T^2} \ , \\ u^\mu u^\nu \Delta^{\rho\sigma} \partial_\rho \beta_\sigma &= u^\mu u^\nu \frac{\theta}{T} \ , \\ \Delta^{\mu\nu} u^\rho u^\sigma \partial_\rho \beta_\sigma &= -\Delta^{\mu\nu} \frac{\dot{T}}{T^2} \ , \\ \Delta^{\mu\nu} \Delta^{\rho\sigma} \partial_\rho \beta_\sigma &= \Delta^{\mu\nu} \frac{\theta}{T} \ , \\ (u^\mu u^\rho \Delta^{\nu\sigma} + u^\mu u^\sigma \Delta^{\nu\rho} + \mu \leftrightarrow \nu)\, \partial_\rho \beta_\sigma &= u^\mu \left(\frac{A^\nu}{T} - \frac{\partial_\perp^\nu T}{T^2}\right) + u^\nu \left(\frac{A^\mu}{T} - \frac{\partial_\perp^\mu T}{T^2}\right) \ , \\ (\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho})\, \partial_\rho \beta_\sigma &= \frac{1}{T} \left(\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu\right) \ . \end{aligned} \tag{100}$$

For the tensor Y the three possible structures are

$$\begin{aligned} u^\mu u^\nu u^\sigma \partial_\sigma \zeta &= u^\mu u^\nu \dot{\zeta} \ , \\ \Delta^{\mu\nu} u^\sigma \partial_\sigma \zeta &= \Delta^{\mu\nu} \dot{\zeta} \ , \\ (u^\mu \Delta^{\nu\sigma} + u^\nu \Delta^{\mu\sigma})\, \partial_\sigma \zeta &= u^\mu \partial_\perp^\nu \zeta + u^\nu \partial_\perp^\mu \zeta \ . \end{aligned} \tag{101}$$

All together this leads to

$$\begin{aligned}\Delta_{\rm diss}T^{\mu\nu} &= -X_1 u^\mu u^\nu \frac{\dot T}{T^2} + X_2 u^\mu u^\nu \frac{\theta}{T} - X_3 \Delta^{\mu\nu}\frac{\dot T}{T^2} + X_4 \Delta^{\mu\nu}\frac{\theta}{T} \\ &+ X_5\left[u^\mu\left(\frac{A^\nu}{T} - \frac{\partial_\perp^\nu T}{T^2}\right) + u^\nu\left(\frac{A^\mu}{T} - \frac{\partial_\perp^\mu T}{T^2}\right)\right] \\ &+ \frac{X_6}{T}\left(\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu\right) - Y_1 u^\mu u^\nu \dot\zeta - Y_2 \Delta^{\mu\nu}\dot\zeta \\ &- Y_3\left(u^\mu \partial_\perp^\nu \zeta + u^\nu \partial_\perp^\mu \zeta\right) \ .\end{aligned} \tag{102}$$

Now we want to relate the scalar coefficients X_i and Y_i to the usual dissipative structures. We start from the decomposition

$$T^{\mu\nu} = T^{\mu\nu}_{\rm id} + \Delta_{\rm diss}T^{\mu\nu} \ , \tag{103}$$

where

$$T^{\mu\nu}_{\rm id} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} \ . \tag{104}$$

The dissipative correction is decomposed as

$$\Delta_{\rm diss}T^{\mu\nu} = \Delta\varepsilon\, u^\mu u^\nu + \mathfrak{q}^\mu u^\nu + \mathfrak{q}^\nu u^\mu - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \ , \tag{105}$$

where $\mathfrak{q}^\mu$ is the heat flow, Π is the non-equilibrium pressure and $\pi^{\mu\nu}$ is the shear-stress tensor. The latter is symmetric, transverse and traceless.

Using

$$\sigma^{\mu\nu} \equiv \frac{1}{2}\left(\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\theta \ , \tag{106}$$

one may rewrite the last tensor structure as

$$\frac{X_6}{T}\left(\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu\right) = \frac{2X_6}{T}\sigma^{\mu\nu} + \Delta^{\mu\nu}\frac{2X_6}{3T}\theta \ . \tag{107}$$

Therefore the off-shell identifications are

$$\begin{aligned}\Delta\varepsilon &= -X_1\frac{\dot T}{T^2} + X_2\frac{\theta}{T} - Y_1\dot\zeta \ , \\ -\Pi &= -X_3\frac{\dot T}{T^2} + X_4\frac{\theta}{T} - Y_2\dot\zeta + \frac{2X_6}{3T}\theta \ , \\ \mathfrak{q}^\mu &= X_5\left(\frac{A^\mu}{T} - \frac{\partial_\perp^\mu T}{T^2}\right) - Y_3\partial_\perp^\mu\zeta \ , \\ \pi^{\mu\nu} &= \frac{2X_6}{T}\sigma^{\mu\nu} \ .\end{aligned} \tag{108}$$

The same procedure can be developed for the four-current:

$$\Delta_{\rm diss}j^\mu(x) = Z^{\mu,\,\rho\sigma}\partial_\rho\beta_\sigma(x) - W^{\mu,\,\sigma}\partial_\sigma\zeta(x) \ . \tag{109}$$

The most general decomposition compatible with isotropy gives

$$\begin{aligned}\Delta_{\rm diss}j^\mu &= Z_1 u^\mu u^\rho u^\sigma \partial_\rho\beta_\sigma + Z_2 u^\mu \Delta^{\rho\sigma}\partial_\rho\beta_\sigma + Z_3\left(u^\rho\Delta^{\mu\sigma} + u^\sigma\Delta^{\mu\rho}\right)\partial_\rho\beta_\sigma \\ &\quad - W_1 u^\mu u^\sigma\partial_\sigma\zeta - W_2\Delta^{\mu\sigma}\partial_\sigma\zeta \\ &= u^\mu\left(-Z_1\frac{\dot T}{T^2} + Z_2\frac{\theta}{T} - W_1\dot\zeta\right) \\ &\quad + Z_3\left(\frac{A^\mu}{T} - \frac{\partial_\perp^\mu T}{T^2}\right) - W_2\partial_\perp^\mu\zeta \ .\end{aligned} \tag{110}$$

Expanding the dissipative correction to the current as

$$j^\mu = j^\mu_{\rm id} + \Delta_{\rm diss} j^\mu = n u^\mu + \Delta n\, u^\mu + \mathcal{V}^\mu \,, \tag{111}$$

one obtains

$$\begin{aligned} \Delta n &= -Z_1 \frac{\dot T}{T^2} + Z_2 \frac{\theta}{T} - W_1 \dot\zeta \,, \\ \mathcal{V}^\mu &= Z_3 \left(\frac{A^\mu}{T} - \frac{\partial^\mu_\perp T}{T^2} \right) - W_2 \partial^\mu_\perp \zeta \,. \end{aligned} \tag{112}$$

The previous equations are still off-shell identifications. They tell us how the most general first-order corrections decompose into hydrodynamic irreducible structures, but they do not yet give the usual transport coefficients. To obtain them one proceeds exactly as in the scalar case: one uses the ideal equations of motion to eliminate the time derivatives $\dot T$ and $\dot\zeta$ in favor of θ, and the transverse ideal Euler equation to eliminate the acceleration.

The scalar part is reduced by the same relations derived above,

$$\dot T = -\mathcal{A}\theta, \qquad \dot\zeta = -\mathcal{B}\theta \,. \tag{113}$$

Therefore

$$\Delta\varepsilon = \left(\frac{X_1 \mathcal{A}}{T^2} + \frac{X_2}{T} + Y_1 \mathcal{B} \right) \theta \,, \tag{114}$$

and

$$\Delta n = \left(\frac{Z_1 \mathcal{A}}{T^2} + \frac{Z_2}{T} + W_1 \mathcal{B} \right) \theta \,. \tag{115}$$

These two quantities are scalar dissipative corrections to the local energy and charge densities. Their precise value depends on the hydrodynamic frame. For example, in the Landau frame one chooses the local velocity and temperature so that $\Delta\varepsilon = 0$, while in the Eckart frame one chooses the velocity so that the dissipative charge diffusion current vanishes.

The bulk viscous pressure is defined by

$$\Pi = -\zeta_V \theta \,. \tag{116}$$

Since the coefficient multiplying $\Delta^{\mu\nu}$ in $\Delta_{\rm diss} T^{\mu\nu}$ is $-\Pi$, one obtains

$$\zeta_V = \frac{X_3 \mathcal{A}}{T^2} + \frac{X_4}{T} + Y_2 \mathcal{B} + \frac{2X_6}{3T} \,. \tag{117}$$

This is the on-shell bulk-viscosity coefficient in the present notation.

The traceless tensor part does not require any use of the equations of motion. Comparing

$$\pi^{\mu\nu} = \frac{2X_6}{T} \sigma^{\mu\nu} \tag{118}$$

with the standard constitutive relation

$$\pi^{\mu\nu} = 2\eta\, \sigma^{\mu\nu} \,, \tag{119}$$

one immediately finds

$$\eta = \frac{X_6}{T} \,. \tag{120}$$

This identification is frame-independent, because the shear tensor is transverse and traceless and cannot mix with scalar or vector redefinitions of the hydrodynamic fields.

Finally, the vector sector is reduced by using the transverse part of the ideal conservation equation,

$$(\varepsilon + p) A^\mu = \partial^\mu_\perp p \,. \tag{121}$$

Using

$$dp = s\,dT + n\,d\mu\ , \qquad \mu = T\zeta\ , \tag{122}$$

one obtains

$$dp = \frac{\varepsilon + p}{T} dT + nT\,d\zeta\ . \tag{123}$$

Hence

$$\frac{A^\mu}{T} - \frac{\partial_\perp^\mu T}{T^2} = \frac{n}{\varepsilon + p}\partial_\perp^\mu \zeta\ . \tag{124}$$

The heat flow and charge diffusion current therefore become

$$\mathfrak{q}^\mu = \left(\frac{nX_5}{\varepsilon + p} - Y_3\right)\partial_\perp^\mu \zeta\ , \tag{125}$$

and

$$\mathcal{V}^\mu = \left(\frac{nZ_3}{\varepsilon + p} - W_2\right)\partial_\perp^\mu \zeta\ . \tag{126}$$

Thus the physical vector transport coefficient is again an on-shell combination of microscopic coefficients. In the Landau frame one imposes $\mathfrak{q}^\mu = 0$ and interprets the remaining transverse current as charge diffusion. In the Eckart frame one imposes $\mathcal{V}^\mu = 0$ and the dissipative vector is instead the heat flow.

Summarizing, the on-shell dissipative coefficients obtained from the Kubo coefficients are

$$\eta = \frac{X_6}{T}\ , \tag{127}$$

for the shear viscosity,

$$\zeta_V = \frac{X_3\mathcal{A}}{T^2} + \frac{X_4}{T} + Y_2\mathcal{B} + \frac{2X_6}{3T}\ , \tag{128}$$

for the bulk viscosity, and

$$\mathcal{K}_q = \frac{nX_5}{\varepsilon + p} - Y_3\ , \qquad \mathcal{K}_V = \frac{nZ_3}{\varepsilon + p} - W_2\ , \tag{129}$$

for the vector sector. The precise name assigned to the vector coefficient depends on the hydrodynamic frame convention. The important point is that the microscopic Kubo formulae give the off-shell coefficients X_i, Y_i, Z_i, W_i, while the usual hydrodynamic dissipative coefficients are the on-shell combinations selected by the conservation equations and by the chosen hydrodynamic frame.

IV. EXERCISES

A. Exercise 1

In this exercise we show that the operator

$$\widehat{\rho} = \frac{1}{Z} \mathrm{e}^{-\lambda \widehat{A}} \tag{130}$$

maximizes the entropy functional under the constraint that the average value $\langle \widehat{A} \rangle$ is fixed.

The functional we have to maximize is

$$F[\,\widehat{\rho}\,] = -\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho}) - c\,(\mathrm{Tr}\widehat{\rho} - 1) - \lambda \left(\mathrm{Tr}(\widehat{\rho}\,\widehat{A}) - \langle \widehat{A} \rangle \right) \,. \tag{131}$$

Here c and λ are Lagrange multipliers. The first constraint imposes the normalization of the density operator, while the second one fixes the expectation value of $\widehat{A}$:

$$\mathrm{Tr}\widehat{\rho} = 1 \,, \qquad \mathrm{Tr}(\widehat{\rho}\,\widehat{A}) = \langle \widehat{A} \rangle \,.$$

At the maximum, the functional is stationary, namely

$$\frac{\delta F[\,\widehat{\rho}\,]}{\delta \widehat{\rho}} = 0 \,.$$

Let us now introduce the operator

$$\widehat{X} \equiv \widehat{\mathrm{I}} - \widehat{\rho} \,. \tag{132}$$

This rewriting is useful because it allows us to compute the variation of the functional by considering the change

$$\widehat{X} \longrightarrow \widehat{X} + \delta \widehat{X} \,.$$

Since $\widehat{\rho} = \widehat{\mathrm{I}} - \widehat{X}$, this corresponds to

$$\widehat{\rho} \longrightarrow \widehat{\rho} - \delta \widehat{X} \,.$$

In terms of $\widehat{X}$, the functional (131) becomes

$$F[\,\widehat{X}\,] = -\mathrm{Tr}\left[(\widehat{\mathrm{I}} - \widehat{X}) \ln(\widehat{\mathrm{I}} - \widehat{X}) \right] - c \left[\mathrm{Tr}(\widehat{\mathrm{I}} - \widehat{X}) - 1 \right] - \lambda \left[\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X})\widehat{A} \right) - \langle \widehat{A} \rangle \right] \,. \tag{133}$$

Equivalently, using the linearity and cyclicity of the trace, the last term can also be written as

$$\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X})\widehat{A} \right) = \mathrm{Tr}\widehat{A} - \mathrm{Tr}(\widehat{X}\widehat{A}) \,. \tag{134}$$

We now compute the finite difference

$$F[\,\widehat{X} + \delta\widehat{X}\,] - F[\,\widehat{X}\,] \tag{135}$$

to first order in $\delta\widehat{X}$. Explicitly,

$$\begin{aligned} F[\,\widehat{X} + \delta\widehat{X}\,] - F[\,\widehat{X}\,] = & -\mathrm{Tr}\left[(\widehat{\mathrm{I}} - \widehat{X} - \delta\widehat{X}) \ln(\widehat{\mathrm{I}} - \widehat{X} - \delta\widehat{X}) \right] \\ & + \mathrm{Tr}\left[(\widehat{\mathrm{I}} - \widehat{X}) \ln(\widehat{\mathrm{I}} - \widehat{X}) \right] \\ & - c \left[\mathrm{Tr}(\widehat{\mathrm{I}} - \widehat{X} - \delta\widehat{X}) - 1 \right] + c \left[\mathrm{Tr}(\widehat{\mathrm{I}} - \widehat{X}) - 1 \right] \\ & - \lambda \left[\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X} - \delta\widehat{X})\widehat{A} \right) - \langle \widehat{A} \rangle \right] \\ & + \lambda \left[\mathrm{Tr}\left((\widehat{\mathrm{I}} - \widehat{X})\widehat{A} \right) - \langle \widehat{A} \rangle \right] \,. \end{aligned} \tag{136}$$

Let us first consider the entropy term. Up to first order in $\delta\widehat{X}$,

$$
\begin{aligned}
&-\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})\right]+\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X})\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\\
&=\operatorname{Tr}\left[\delta\widehat{X}\,\ln(\widehat{\mathrm{I}}-\widehat{X})\right]-\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}+\mathcal{O}(\delta\widehat{X}^2)\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
F[\,\widehat{X}+\delta\widehat{X}\,]-F[\,\widehat{X}\,]=\operatorname{Tr}\left[\delta\widehat{X}\,\ln(\widehat{\mathrm{I}}-\widehat{X})\right]-\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}\\
+c\operatorname{Tr}(\delta\widehat{X})+\lambda\operatorname{Tr}(\delta\widehat{X}\,\widehat{A})+\mathcal{O}(\delta\widehat{X}^2)\,.
\end{aligned}
$$

We now have to compute the variation of the logarithm. The logarithm of an operator can be formally defined through its Taylor expansion. In particular, for sufficiently small $\widehat{X}$ one has

$$
\ln(\widehat{\mathrm{I}}-\widehat{X})=-\sum_{n=1}^{\infty}\frac{\widehat{X}^n}{n}\,. \tag{137}
$$

Hence:

$$
\begin{aligned}
&\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\\
&=-\sum_{n=1}^{\infty}\frac{(\widehat{X}+\delta\widehat{X})^n-\widehat{X}^n}{n}\,.
\end{aligned}
$$

Since, in general, $\widehat{X}$ and $\delta\widehat{X}$ do not commute, we have to keep track of all possible positions of $\delta\widehat{X}$. To first order,

$$
(\widehat{X}+\delta\widehat{X})^n=\widehat{X}^n+\widehat{X}^{n-1}\delta\widehat{X}+\widehat{X}^{n-2}\delta\widehat{X}\,\widehat{X}+\cdots+\delta\widehat{X}\,\widehat{X}^{n-1}+\mathcal{O}(\delta\widehat{X}^2)\,.
$$

Therefore,

$$
\begin{aligned}
&\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\\
&=-\sum_{n=1}^{\infty}\frac{\widehat{X}^{n-1}\delta\widehat{X}+\widehat{X}^{n-2}\delta\widehat{X}\,\widehat{X}+\cdots+\delta\widehat{X}\,\widehat{X}^{n-1}}{n}+\mathcal{O}(\delta\widehat{X}^2)\,.
\end{aligned}
$$

Now we plugging this in the variation of the entropy and using the cyclic property of the trace we find that all terms contribute in the same way:

$$
\begin{aligned}
&\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}\\
&=-\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X})\delta\widehat{X}\sum_{n=1}^{\infty}\widehat{X}^{n-1}\right]+\mathcal{O}(\delta\widehat{X}^2)\,.
\end{aligned}
$$

Since the sum reduces to a geometric series:

$$
\sum_{n=1}^{\infty}\widehat{X}^{n-1}=(\widehat{\mathrm{I}}-\widehat{X})^{-1}\,, \tag{138}
$$

we obtain

$$
\begin{aligned}
&\operatorname{Tr}\left\{(\widehat{\mathrm{I}}-\widehat{X})\left[\ln(\widehat{\mathrm{I}}-\widehat{X}-\delta\widehat{X})-\ln(\widehat{\mathrm{I}}-\widehat{X})\right]\right\}\\
&=-\operatorname{Tr}\left[(\widehat{\mathrm{I}}-\widehat{X})\delta\widehat{X}(\widehat{\mathrm{I}}-\widehat{X})^{-1}\right]+\mathcal{O}(\delta\widehat{X}^2)\\
&=-\operatorname{Tr}(\delta\widehat{X})+\mathcal{O}(\delta\widehat{X}^2)\,.
\end{aligned}
$$

Therefore the first-order variation of the functional is

$$
F[\,\widehat{X}+\delta\widehat{X}\,]-F[\,\widehat{X}\,]=\operatorname{Tr}\left[\delta\widehat{X}\,\ln(\widehat{\mathrm{I}}-\widehat{X})\right]+\operatorname{Tr}(\delta\widehat{X})+c\operatorname{Tr}(\delta\widehat{X})+\lambda\operatorname{Tr}(\delta\widehat{X}\,\widehat{A})+\mathcal{O}(\delta\widehat{X}^2)\,,
$$

or equivalently:

$$F[\,\widehat{X}+\delta\widehat{X}\,]-F[\,\widehat{X}\,]=\mathrm{Tr}\left\{\delta\widehat{X}\left[\ln(\widehat{\mathrm{I}}-\widehat{X})+\widehat{\mathrm{I}}+c\widehat{\mathrm{I}}+\lambda\widehat{A}\right]\right\}+\mathcal{O}(\delta\widehat{X}^2)\,. \tag{139}$$

We can now rewrite this result in terms of $\widehat{\rho}$ for which

$$\widehat{\rho}=\widehat{\mathrm{I}}-\widehat{X}\implies\delta\widehat{\rho}=-\delta\widehat{X}\,.$$

Then:

$$F[\,\widehat{\rho}+\delta\widehat{\rho}]-F[\,\widehat{\rho}]=-\mathrm{Tr}\left\{\delta\widehat{\rho}\left[\ln\widehat{\rho}+\widehat{\mathrm{I}}+c\widehat{\mathrm{I}}+\lambda\widehat{A}\right]\right\}+\mathcal{O}(\delta\widehat{\rho}^2)\,. \tag{140}$$

At the stationary point, the first-order variation must vanish for arbitrary Hermitian variations $\delta\widehat{\rho}$. Therefore,

$$\ln\widehat{\rho}+\widehat{\mathrm{I}}+c\widehat{\mathrm{I}}+\lambda\widehat{A}=0\,,$$

which implies

$$\ln\widehat{\rho}=-(1+c)\widehat{\mathrm{I}}-\lambda\widehat{A}\,.$$

Exponentiating both sides, and using the fact that the identity operator commutes with $\widehat{A}$, we obtain

$$\widehat{\rho}=\mathrm{e}^{-(1+c)\widehat{\mathrm{I}}-\lambda\widehat{A}}=\mathrm{e}^{-(1+c)}\mathrm{e}^{-\lambda\widehat{A}}\,. \tag{141}$$

The constant $\mathrm{e}^{-(1+c)}$ is fixed by the normalization condition $\mathrm{Tr}\widehat{\rho}=1$. Therefore,

$$1=\mathrm{Tr}\widehat{\rho}=\mathrm{e}^{-(1+c)}\mathrm{Tr}\left(\mathrm{e}^{-\lambda\widehat{A}}\right)\,.$$

Defining the partition function as

$$Z\equiv\mathrm{Tr}\left(\mathrm{e}^{-\lambda\widehat{A}}\right)\,, \tag{142}$$

and identifying $Z=\mathrm{e}^{-(1+c)}$ we finally obtain:

$$\widehat{\rho}=\frac{1}{Z}\mathrm{e}^{-\lambda\widehat{A}}\,. \tag{143}$$

The Lagrange multiplier λ is fixed by the condition

$$\langle\widehat{A}\rangle=\mathrm{Tr}(\widehat{\rho}\,\widehat{A})\,. \tag{144}$$

In terms of the partition function, this relation can also be written as

$$\langle\widehat{A}\rangle=-\frac{\partial\ln Z}{\partial\lambda}\,. \tag{145}$$

Finally, since the von Neumann entropy is a concave functional of $\widehat{\rho}$ and the constraints are linear in $\widehat{\rho}$, the stationary point found above is the maximum of the entropy under the imposed constraints.

Now lets specialize to the canonical case. Note that, as long as they are commuting, we can simply replace $\widehat{A}$ with any set of operators we want each of one with its own Lagrange multiplier:

$$\widehat{\rho}=\frac{1}{Z}\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}\,,\qquad Z=\mathrm{Tr}\left(\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}\right)\,.$$

Then is easy to prove that:

$$\langle\widehat{P}^{\nu}\rangle=-\frac{\partial\ln Z}{\partial\beta_\nu}\,,\qquad\langle\widehat{Q}\rangle=\frac{\partial\ln Z}{\partial\zeta}$$

B. Exercise 2

For a relativistic system, the possible global-equilibrium configurations are richer than in the non-relativistic case. Besides homogeneous equilibrium, one can have equilibrium states with non-vanishing thermal vorticity. In the simplest situations, these configurations can be organized into two basic classes: global equilibrium with rotation and global equilibrium with acceleration. The first one has a direct classical analogue, namely a rigidly rotating body in thermal equilibrium. The second one is intrinsically relativistic and is associated with Lorentz boosts.

Let us start from the operator maximizing the entropy,

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) \right] . \tag{146}$$

The condition for global equilibrium is that the operator be independent of the integration hypersurface Σ. This means that the current appearing in the exponent must be conserved. Therefore, we require

$$\nabla_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) = 0 . \tag{147}$$

Expanding the derivative, we obtain

$$\nabla_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) = \beta_\nu \nabla_\mu \widehat{T}^{\mu\nu} + \widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu - \zeta \nabla_\mu \widehat{j}^\mu - \widehat{j}^\mu \nabla_\mu \zeta .$$

Using the conservation equations

$$\nabla_\mu \widehat{T}^{\mu\nu} = 0 , \qquad \nabla_\mu \widehat{j}^\mu = 0 ,$$

the condition reduces to

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta = 0 . \tag{148}$$

Since the stress-energy tensor and the conserved current are independent hydrodynamic operators, the global-equilibrium condition requires separately

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu = 0 , \qquad \widehat{j}^\mu \nabla_\mu \zeta = 0 .$$

The second equation implies that the reduced chemical potential ζ must be constant:

$$\nabla_\mu \zeta = 0 .$$

For the four-temperature, we use the symmetry of the stress-energy tensor. Since $\widehat{T}^{\mu\nu} = \widehat{T}^{\nu\mu}$, we can write

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu = \frac{1}{2} \widehat{T}^{\mu\nu} \left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu \right) + \frac{1}{2} \widehat{T}^{\mu\nu} \left(\nabla_\mu \beta_\nu - \nabla_\nu \beta_\mu \right) .$$

The second term vanishes because it is the contraction of a symmetric tensor with an antisymmetric tensor. Thus,

$$\widehat{T}^{\mu\nu} \nabla_\mu \beta_\nu = \frac{1}{2} \widehat{T}^{\mu\nu} \left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu \right) .$$

For this expression to vanish in general, the four-temperature must satisfy

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 . \tag{149}$$

This is the Killing equation. Therefore, at global equilibrium, the four-temperature β^μ must be a Killing vector.

A Killing vector has a clear geometrical meaning: it is the generator of an isometry of spacetime. In other words, the flow generated by a Killing vector leaves the metric invariant. This can be written as

$$\mathfrak{L}_\beta g_{\mu\nu} = 0 ,$$

where $\mathfrak{L}_\beta$ denotes the Lie derivative along β^μ. Therefore, the field lines of β^μ are the curves along which the geometry is unchanged. This is the geometrical reason why a thermal state can be in equilibrium even when β^μ depends explicitly on the spacetime point: what matters is not ordinary homogeneity, but stationarity with respect to the flow

generated by β^μ. The fact that β^μ is a Killing vector is not yet enough to guarantee a thermodynamic interpretation. In order to describe a physical thermal state, the four-temperature has to be timelike and future-directed:

$$\beta^2 > 0 \,, \qquad \beta^0 > 0 \,.$$

Only in the region where these conditions hold can we write

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)} \,, \qquad T(x) = \frac{1}{\sqrt{\beta^2(x)}} \,, \qquad u^\mu(x) = \frac{\beta^\mu(x)}{\sqrt{\beta^2(x)}} \,. \tag{150}$$

Here u^μ is the local four-velocity of the fluid and T is the proper temperature, namely the temperature measured by a thermometer comoving with the fluid element.

An important consequence of the Killing equation is that the proper temperature is constant along the flow lines of β^μ. To see this, we contract the Killing equation (149) with $\beta^\mu\beta^\nu$:

$$\beta^\mu\beta^\nu \left(\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu\right) = 0 \,.$$

The two terms are equal, and therefore

$$2\beta^\mu\beta^\nu\nabla_\mu\beta_\nu = 0 \,.$$

On the other hand,

$$\beta^\mu\nabla_\mu(\beta^2) = \beta^\mu\nabla_\mu(\beta^\nu\beta_\nu) = 2\beta^\mu\beta^\nu\nabla_\mu\beta_\nu \,.$$

Thus

$$\beta^\mu\nabla_\mu(\beta^2) = 0 \,. \tag{151}$$

Using $T = 1/\sqrt{\beta^2}$, equation (151) implies

$$\beta^\mu\nabla_\mu T = 0 \,.$$

Since u^μ is proportional to β^μ, we also have

$$u^\mu\nabla_\mu T = 0 \,.$$

Therefore, a comoving observer measures a constant temperature along its own worldline. However, different flow lines can have different values of the proper temperature. This point is crucial for understanding both rotating and accelerated equilibrium.

We now solve the equation for the particular case of Minkowski space-time. First, for Minkowski, the Christoffel are vanishing and the covariant derivative is simply reduced to the partial one. We then must solve the following set of equations:

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0 \,. \tag{152}$$

The solution of this equation is quite simple. Take the derivative with respect to ∂_λ:

$$\partial_\lambda\partial_\mu\beta_\nu + \partial_\lambda\partial_\nu\beta_\mu = 0 \,.$$

Define the tensor $B_{\lambda\mu\nu} \equiv \partial_\lambda\partial_\mu\beta_\nu$. Due to the Killing equation (152) the tensor is antisymmetric in the last two indeces whereas due to the commutativity of partial derivatives it is symmetric in the first two:

$$B_{\lambda\mu\nu} = B_{\mu\lambda\nu} \,, \qquad B_{\lambda\mu\nu} = -B_{\lambda\nu\mu} \,. \tag{153}$$

But this implies:

$$B_{\lambda\mu\nu} = -B_{\lambda\nu\mu} = -B_{\nu\lambda\mu} = B_{\nu\mu\lambda} = B_{\mu\nu\lambda} = -B_{\mu\lambda\nu} = -B_{\lambda\mu\nu} \,,$$

i.e. $B_{\lambda\mu\nu} = -B_{\lambda\mu\nu} = 0$:

$$\partial_\lambda\partial_\mu\beta_\nu = 0 \,. \tag{154}$$

The four-temperature cannot depend more than linearly from x:

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \ , \tag{155}$$

where both b and ϖ are constant. In particular the latter has to be antisymmetric due to the Killing equation (152):

$$\partial_\mu \beta_\nu = \varpi_{\mu\nu} = -\partial_{\nu\mu} = -\varpi_{\nu\mu} \ ,$$

and one can define the thermal vorticity as:

$$\varpi_{\mu\nu} = \frac{1}{2} \left(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu \right) \ . \tag{156}$$

Then in Minkowski spacetime the general solution of the Killing equation is:

$$\beta_\mu(x) = b_\mu + \varpi_{\mu\nu} x^\nu \ , \tag{157}$$

The four constants contained in b_μ are associated with translations, while the six independent constants contained in $\varpi_{\mu\nu}$ are associated with Lorentz transformations, namely rotations and boosts. Thus the most general Killing vector in Minkowski spacetime is determined by ten constants, corresponding to the ten generators of the Poincare group.

We can show how the thermal vorticity encodes both the effects of acceleration and rotation. Indeed any antisymmetric rank two tensor can be decomposed in two space-like four-vectors. A generic space-like four vector is orthogonal to a given time-like four-vector. We have at disposal the four-velocity so we can define:

$$\alpha^\mu = \varpi^{\mu\nu} u_\nu \ , \qquad w^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu \varpi_{\rho\sigma} \ . \tag{158}$$

Due to the antisymmetricity of ϖ and ϵ it follows:

$$\alpha^\mu u_\mu = 0 \ , \quad w^\mu u_\mu = 0 \ , \tag{159}$$

and:

$$\varpi_{\mu\nu} = \alpha_\mu u_\nu - \alpha_\mu u_\nu + \epsilon_{\mu\nu\rho\sigma} w^\rho u^\sigma \ . \tag{160}$$

Now, component wise, we have:

$$\varpi_{0i} = \alpha^i \ , \quad \varpi_{ij} = \epsilon_{ijk} w^k \ ,$$

that is:

$$\varpi_{\mu\nu} \sim \begin{pmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ -\alpha_x & 0 & w_z & -w_y \\ -\alpha_y & -w_z & 0 & w_x \\ -\alpha_z & w_y & -w_x & 0 \end{pmatrix} \ . \tag{161}$$

Now we can clearly see how α and w are directly related with the four-acceleration vector A and the kinetic vorticity ω defined as:

$$A^\mu = u^\nu \partial_\nu u^\mu \ , \qquad \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \ . \tag{162}$$

We start from the four-acceleration. Given that $u_\mu = T \beta_\mu$ we have:

$$A^\mu = u^\nu \partial_\nu (T \beta^\mu) = \beta^\mu u^\nu \partial_\nu T + u^\nu T \partial_\nu \beta^\mu \ .$$

The first term vanishes given that the temperature is constant along the flow lines while for the second one we can replace the gradient with the vorticity using the solution (157):

$$A^\mu = T u^\nu \varpi^\mu_\nu = T \alpha^\mu \ ,$$

so that:

$$\alpha^\mu = \frac{A^\mu}{T} \ . \tag{163}$$

Now lets consider w^μ. We get:

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho (T\beta_\sigma) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \left(\beta_\sigma \partial_\rho T + T\partial_\rho \beta_\sigma\right) \ .$$

Being $u_\nu = T\beta_\nu$ the first piece is proportional to $\epsilon^{\mu\nu\rho\sigma}\beta_\nu\beta_\rho = 0$ which is vanishing due to antisimmetricity. The second term instead is reletated with the thermal vorticity (156):

$$\partial_\rho \beta_\sigma = \frac{1}{2}(\partial_\rho \beta_\sigma - \partial_\sigma \beta_\rho) + \frac{1}{2}(\partial_\rho \beta_\sigma + \partial_\sigma \beta_\rho) = \varpi_{\rho\sigma} \ ,$$

where the symmetric part vanishes due to (152), so that:

$$\omega^\mu = \frac{T}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \varpi_{\rho\sigma} = T w^\mu \ ,$$

that is:

$$w^\mu = \frac{\omega^\mu}{T} \ . \tag{164}$$

We now return to the particular case of Minkowski spacetime. From (157), one sees that when $\varpi_{\mu\nu} \neq 0$, the four-temperature depends explicitly on the spacetime coordinate x^ν. Therefore, in a generic inertial frame, the equilibrium state is not homogeneous and, in some cases, it is not even time independent with respect to the inertial time coordinate. Nevertheless, it is still a global-equilibrium state because it is stationary with respect to the flow generated by the Killing vector β^μ.

In the following, we discuss two important examples: global equilibrium with rotation and global equilibrium with acceleration.

C. Exercise 3

Let us start with the rotating case. We choose the rotation axis to be the z axis. The corresponding parameters are

$$b_\mu = \frac{1}{T_0}(1, \mathbf{0}) \ , \qquad \varpi_{\mu\nu} = \frac{\omega}{T_0}\left(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu}\right) \ .$$

Here T_0 is not the proper temperature at a generic point of the rotating system. It is the temperature measured on the rotation axis, where the local velocity of the rotating medium vanishes.

Substituting this choice into the global-equilibrium statistical operator gives

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\frac{\widehat{H}}{T_0} + \frac{\omega}{T_0}\widehat{J}_z\right) \ ,$$

where $\widehat{J}_z = \widehat{J}^{12}$ is the generator of rotations around the z axis. This is the relativistic version of the density operator for a system in rigid rotation: the Hamiltonian is shifted by the angular-momentum generator, weighted by the angular velocity.

For a generic rotation axis, one can write

$$b_\mu = \frac{1}{T_0}(1, \mathbf{0}) \ , \qquad \varpi \cdot x = (0, \boldsymbol{\omega} \times \mathbf{x}) \ ,$$

and the replacement

$$\omega \widehat{J}_z \mapsto \boldsymbol{\omega} \cdot \widehat{\mathbf{J}}$$

has to be made.

The corresponding four-temperature field is

$$\beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x}) \ . \tag{165}$$

For rotation around the z axis,

$$\boldsymbol{\omega} = \omega \hat{\mathbf{z}} \ ,$$

so that

$$\boldsymbol{\omega} \times \mathbf{x} = (-\omega y, \omega x, 0)$$

up to the conventional orientation of the rotation. The important point is that the spatial part of β^μ is tangent to circles centered around the rotation axis.

The vector (165) is not timelike everywhere. Its norm is

$$\beta^2 = \frac{1}{T_0^2} \left(1 - |\boldsymbol{\omega} \times \mathbf{x}|^2 \right) \ .$$

For rotation around the z axis,

$$|\boldsymbol{\omega} \times \mathbf{x}|^2 = \omega^2 r^2 \ , \qquad r = \sqrt{x^2 + y^2} \ ,$$

and therefore

$$\beta^2 = \frac{1}{T_0^2} \left(1 - \omega^2 r^2 \right) \ .$$

The four-temperature is timelike only if

$$1 - \omega^2 r^2 > 0 \ ,$$

namely

$$r < \frac{1}{|\omega|} \ .$$

The surface

$$r = \frac{1}{|\omega|}$$

is called the light cylinder. At this radius, the tangential velocity of a rigidly rotating fluid element would reach the speed of light. Therefore, a global equilibrium state with rigid rotation cannot be extended beyond the light cylinder.

The proper temperature is obtained from the norm of the four-temperature:

$$T(x) = \frac{1}{\sqrt{\beta^2(x)}} \ .$$

For the rotating configuration, this gives

$$T(r) = \frac{T_0}{\sqrt{1 - \omega^2 r^2}} \ .$$

Thus the proper temperature measured by a comoving observer increases with the distance from the rotation axis. This does not mean that the temperature changes in time for a given fluid element. A fluid element at fixed radius r follows a circular trajectory, and along this trajectory the value of r is constant. Therefore, the proper temperature measured by that fluid element is constant along its motion.

Let us make this geometrical statement more explicit by studying the field lines of β^μ. They are defined by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \beta^\mu(x) \ .$$

Using (165), we get

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{T_0} \ , \qquad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\lambda} = \frac{\boldsymbol{\omega} \times \mathbf{x}}{T_0} \ .$$

Dividing the spatial equation by the time equation gives

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{x} \; .$$

For rotation around the z axis, this becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\omega y \; , \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \omega x \; , \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0 \; .$$

The solutions are circular trajectories:

$$x(t) = r\cos(\omega t + \phi_0) \; , \qquad y(t) = r\sin(\omega t + \phi_0) \; , \qquad z(t) = z_0 \; .$$

Therefore, the field lines of β^μ are helices in spacetime: the spatial motion is circular, while the time coordinate increases monotonically. These are the worldlines of the comoving fluid elements in rigid rotation.

This is the correct physical interpretation of rotating global equilibrium. The state is not spatially homogeneous, because the proper temperature depends on r. Nevertheless, it is in global equilibrium because the system is stationary along the flow generated by β^μ. Each comoving observer measures a constant temperature along its own worldline, but observers at different radii measure different proper temperatures.

The second basic equilibrium configuration is the accelerated one. This configuration is purely relativistic and has no direct non-relativistic analogue. It is associated with Lorentz boosts rather than with ordinary spatial rotations.

We choose the acceleration to be along the z direction and take

$$b_\mu = \frac{1}{T_0}(1, \mathbf{0}) \; , \qquad \varpi_{\mu\nu} = \frac{a}{T_0}\left(g_{0\mu}g_{3\nu} - g_{0\nu}g_{3\mu}\right) \; .$$

The contraction of the thermal vorticity with the Lorentz generators is proportional to the generator of boosts along the z direction. The statistical operator then reads

$$\widehat{\rho}_{\mathrm{GE}} = \frac{1}{Z}\exp\left(-\frac{\widehat{H}}{T_0} + \frac{a}{T_0}\widehat{K}_z\right) \; .$$

This operator is peculiar because it involves the boost generator. The boost generator has an explicit time dependence. With the convention used here, it can be written as

$$\widehat{K}_z = \widehat{J}_{30} = t\widehat{P}_z - \int \mathrm{d}^3x \, z \, \widehat{T}^{00} \; .$$

Nevertheless, the statistical operator is time independent, as required by global equilibrium. The explicit time dependence of $\widehat{K}_z$ is compensated by its non-vanishing commutator with the Hamiltonian. With the corresponding convention for the time evolution of operators, one has

$$\frac{\mathrm{d}\widehat{K}_z}{\mathrm{d}t} = \frac{\partial \widehat{K}_z}{\partial t} - \mathrm{i}\left[\widehat{H}, \widehat{K}_z\right] = \widehat{P}_z - \widehat{P}_z = 0 \; .$$

Therefore, the boost generator is a conserved generator, and the statistical operator is indeed independent of time.

It is important to understand the physical meaning of the parameter a appearing in the statistical operator. The notation may suggest that the whole fluid is accelerating with proper acceleration a, but this is not correct. The parameter a fixes the normalization of the boost Killing field and identifies a special reference hyperbola, namely the one whose proper acceleration is a. However, a spatially extended accelerated fluid in global equilibrium cannot have the same non-zero proper acceleration at every point. Different fluid elements follow different hyperbolic trajectories, and therefore they have different proper accelerations.

With the above choice of parameters, the four-temperature field can be written in a transparent way by introducing

$$z' = z + \frac{1}{a} \; .$$

Then

$$\beta^\mu = \frac{1}{T_0}\left(1 + az, 0, 0, at\right) = \frac{a}{T_0}\left(z', 0, 0, t\right) \; . \tag{166}$$

Notice that here we are writing the contravariant components of β^μ. With the metric convention

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \, ,$$

the covariant components are

$$\beta_\mu = \frac{a}{T_0} \left(z', 0, 0, -t \right) \, .$$

This distinction is important because the sign of the spatial covariant component changes.

The vector field (166) is proportional to the boost Killing vector

$$z' \partial_t + t \partial_z \, .$$

Indeed,

$$\beta = \beta^\mu \partial_\mu = \frac{a}{T_0} \left(z' \partial_t + t \partial_z \right) \, .$$

Thus the accelerated equilibrium state is stationary with respect to a boost flow, not with respect to ordinary inertial time translations. This is the essential difference between this configuration and homogeneous equilibrium.

The norm of the four-temperature is

$$\beta^2 = \frac{a^2}{T_0^2} \left(z'^2 - t^2 \right) \, .$$

Therefore, the proper temperature field is

$$T(t, z) = \frac{1}{\sqrt{\beta^2}} = \frac{T_0}{a \sqrt{z'^2 - t^2}} \, .$$

The four-temperature is timelike only in the region

$$z'^2 - t^2 > 0 \, .$$

This region consists of two Rindler wedges, shifted by the replacement $z \mapsto z' = z + 1/a$. In the right wedge one has

$$z' > |t| \, .$$

The boundaries

$$z' = \pm t$$

are null surfaces. On these surfaces $\beta^2 = 0$, so the Killing vector becomes lightlike. Therefore, the accelerated equilibrium state is physically meaningful only inside the wedge where β^μ is timelike and future-directed.

Let us now study the field lines of (166). They are defined by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \beta^\mu(x) \, .$$

For the accelerated field this gives

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{a}{T_0} z' \, , \qquad \frac{\mathrm{d}z'}{\mathrm{d}\lambda} = \frac{a}{T_0} t \, ,$$

where $z' = z + 1/a$, so $\mathrm{d}z' = \mathrm{d}z$. Dividing the two equations, we find

$$\frac{\mathrm{d}z'}{\mathrm{d}t} = \frac{t}{z'} \, .$$

Hence

$$z' \, \mathrm{d}z' = t \, \mathrm{d}t \, .$$

Integrating, we obtain

$$z'^2 - t^2 = \rho^2 \ ,$$

where ρ is a positive constant labeling the different field lines. Therefore, the field lines of β^μ are hyperbolae in the (t, z) plane.

These hyperbolae are precisely the worldlines of uniformly accelerated observers. A convenient parametrization is

$$t = \rho \sinh \eta \ , \qquad z' = \rho \cosh \eta \ .$$

Along each trajectory,

$$z'^2 - t^2 = \rho^2 = \text{const.}$$

Therefore, the proper temperature measured by the comoving observer on that trajectory is

$$T = \frac{T_0}{a\rho} \ .$$

This is constant along the observer's worldline. Thus, although the field $T(t, z)$ depends on the Minkowski coordinates t and z, a comoving observer does not measure a temperature that changes with time. The apparent time dependence is due to the fact that Minkowski time is not the natural time coordinate adapted to the accelerated equilibrium flow.

The parameter ρ also determines the proper acceleration of the corresponding worldline. The hyperbola

$$z'^2 - t^2 = \rho^2$$

describes a uniformly accelerated observer with proper acceleration

$$\alpha(\rho) = \frac{1}{\rho} \ .$$

Therefore, along a given accelerated trajectory,

$$T(\rho) = \frac{T_0}{a\rho} = \frac{T_0}{a}\alpha(\rho) \ .$$

This formula clarifies the role of a. The parameter a is not the proper acceleration of every fluid element. Rather, it selects the particular hyperbola

$$\rho = \frac{1}{a} \ .$$

On this hyperbola,

$$\alpha = \frac{1}{\rho} = a \ ,$$

and the proper temperature is

$$T = \frac{T_0}{a(1/a)} = T_0 \ .$$

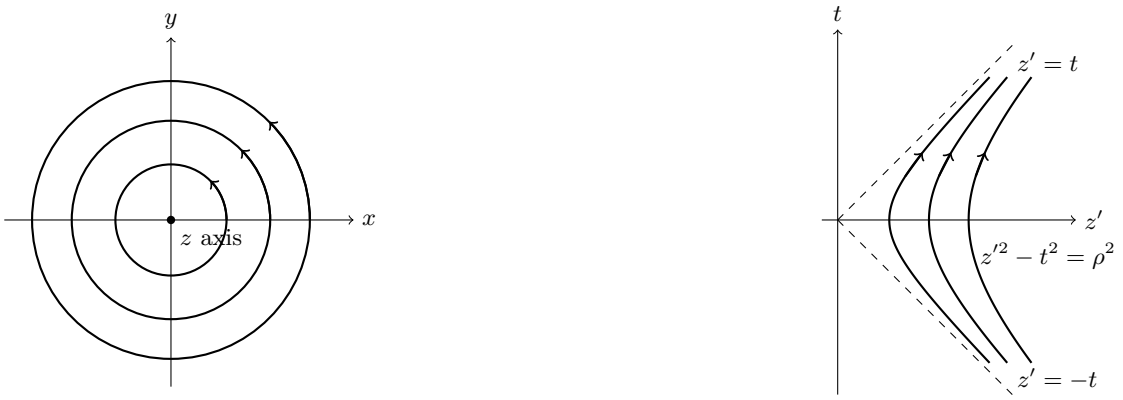
Thus T_0 is the proper temperature measured by the comoving observer whose proper acceleration is a. Other comoving observers, corresponding to other values of ρ, measure different constant temperatures.

This point is a common source of confusion. The statistical operator does not describe a macroscopic fluid whose every element has the same proper acceleration a. Such a configuration would not correspond to a spatially extended equilibrium fluid. Indeed, a fixed value of the proper acceleration,

$$\alpha = \text{const.} \neq 0 \ ,$$

corresponds to a single hyperbola,

$$z'^2 - t^2 = \frac{1}{\alpha^2} \ .$$



Rotating equilibrium
field lines of β^μ are circular

Accelerated equilibrium
field lines of β^μ are hyperbolae

FIG. 2: Schematic comparison between the two basic relativistic global-equilibrium configurations. For rigid rotation, the four-temperature field lines wind around the rotation axis. For accelerated equilibrium, the field lines are the orbits of a boost Killing vector and are hyperbolae in the (t, z') plane.

A single hyperbola is one worldline, not a macroscopic fluid filling a spatial region. A spatially extended accelerated fluid must instead be described by a congruence of hyperbolae, labeled by different values of ρ. Since the proper acceleration is $\alpha(\rho) = 1/\rho$, different fluid elements necessarily have different proper accelerations.

This is also the relativistic condition for maintaining a rigid accelerated configuration. If all parts of an extended body had the same non-zero proper acceleration, their distances in the instantaneous comoving frame would not remain constant. In order for an accelerated body or fluid to remain in equilibrium without stretching or compression, different parts must accelerate differently. The boost Killing flow implements exactly this: the fluid elements follow different hyperbolae, and the proper acceleration varies from one hyperbola to another.

We can now introduce coordinates adapted to this accelerated motion. The natural coordinates are Rindler coordinates. In the right Rindler wedge, we define

$$t = \frac{\mathrm{e}^{a\xi}}{a}\sinh(a\tau)\ , \qquad z' = \frac{\mathrm{e}^{a\xi}}{a}\cosh(a\tau)\ . \tag{167}$$

These coordinates automatically satisfy

$$z'^2 - t^2 = \frac{\mathrm{e}^{2a\xi}}{a^2}\ .$$

Thus a curve at fixed ξ is one of the hyperbolic trajectories discussed above. Comparing with $z'^2 - t^2 = \rho^2$, we have

$$\rho = \frac{\mathrm{e}^{a\xi}}{a}\ .$$

Therefore, fixed ξ means fixed proper acceleration,

$$\alpha(\xi) = \frac{1}{\rho} = a\mathrm{e}^{-a\xi}\ .$$

In particular, the observer at $\xi = 0$ has

$$\rho = \frac{1}{a}\ , \qquad \alpha = a\ .$$

Thus the parameter a is the proper acceleration of the reference Rindler observer located at $\xi = 0$. It is not the proper acceleration of the whole fluid.

The Minkowski metric becomes

$$\mathrm{d}s^2 = \mathrm{e}^{2a\xi}\left(\mathrm{d}\tau^2 - \mathrm{d}\xi^2\right) - \mathrm{d}x^2 - \mathrm{d}y^2\ . \tag{168}$$

The coordinate τ is the natural time coordinate for accelerated observers, while ξ labels different accelerated worldlines.

Using (167), one finds

$$\frac{\partial t}{\partial \tau} = \mathrm{e}^{a\xi}\cosh(a\tau) = az' \ , \qquad \frac{\partial z'}{\partial \tau} = \mathrm{e}^{a\xi}\sinh(a\tau) = at \ .$$

Therefore,

$$\partial_\tau = a\left(z'\partial_t + t\partial_z\right) \ .$$

The boost Killing vector is thus

$$z'\partial_t + t\partial_z = \frac{1}{a}\partial_\tau \ .$$

Since the four-temperature field is

$$\beta = \frac{a}{T_0}\left(z'\partial_t + t\partial_z\right) \ ,$$

we obtain

$$\beta = \frac{1}{T_0}\partial_\tau \ .$$

This is the cleanest way to interpret accelerated global equilibrium. The state is stationary with respect to translations in the Rindler time τ, not with respect to translations in the inertial Minkowski time t.

The temperature profile in Rindler coordinates follows immediately. Since

$$z'^2 - t^2 = \frac{\mathrm{e}^{2a\xi}}{a^2} \ ,$$

we get

$$T(\xi) = \frac{T_0}{a\sqrt{z'^2 - t^2}} = T_0\mathrm{e}^{-a\xi} \ .$$

Equivalently, using $\rho = \mathrm{e}^{a\xi}/a$, one can write

$$T(\rho) = \frac{T_0}{a\rho} \ .$$

This expression shows clearly that the proper temperature depends on which accelerated observer we consider, namely on the value of ξ or ρ. However, it does not depend on the Rindler time τ. Therefore each comoving accelerated observer measures a constant temperature.

An inertial observer at fixed Minkowski position z has a different description. In Minkowski coordinates the proper temperature field is

$$T(t,z) = \frac{T_0}{a\sqrt{(z+1/a)^2 - t^2}} \ .$$

Thus an inertial observer at fixed z sees the local proper temperature of the fluid depend on time. This does not contradict equilibrium, because the inertial observer is not comoving with the fluid and is not following the boost Killing flow. At different Minkowski times, the observer samples different fluid elements belonging to different hyperbolae. The equilibrium is stationary only with respect to the Rindler time τ, namely along the accelerated flow.

To summarize, the accelerated equilibrium configuration is generated by a boost Killing vector. Its field lines are hyperbolae in Minkowski spacetime, corresponding to uniformly accelerated comoving observers. The parameter a is the proper acceleration of the reference observer at $\xi = 0$, for which the proper temperature is T_0. Other fluid elements have different proper accelerations and different constant proper temperatures. Therefore, global accelerated equilibrium is not characterized by a uniform proper acceleration, but by stationarity with respect to the boost Killing flow.

The general equilibrium state is given by a general combination of the rotation and acceleration configuration.

D. Exercise 4

From the definition of the local-equilibrium expectation value in linear response theory,

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm LE} = \langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} + \Delta_{\rm LE} T^{\mu\nu}(x) \ . \tag{169}$$

We want to prove that the leading-order term coincides with the ideal-fluid form, namely

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} = \mathcal{A}(\beta^2)\beta^\mu(x)\beta^\nu(x) + \mathcal{B}(\beta^2) g^{\mu\nu} \ , \tag{170}$$

which is equivalent to

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} = (\varepsilon + p) u^\mu(x) u^\nu(x) - p g^{\mu\nu} \ , \tag{171}$$

where ε and p are the energy density and pressure, respectively.

The fundamental property we use to fix the form of an expectation value is the combination of the transformation properties of the operator we are computing and the symmetries of the state, namely of the density operator. First, the homogeneous equilibrium state is invariant under translations, which are represented on the Hilbert space by the operator

$$\widehat{\rm T}_y = {\rm e}^{{\rm i}\widehat{P}\cdot y} \ . \tag{172}$$

Under translations, the stress-energy tensor transforms as a local field:

$$\widehat{\rm T}_y \, \widehat{T}^{\mu\nu}(x) \, \widehat{\rm T}_y^{-1} = \widehat{T}^{\mu\nu}(x+y) \ . \tag{173}$$

Because of the homogeneity of the equilibrium state, we have

$$\widehat{\rm T}_y \, \widehat{\rho}_{\rm GE} \, \widehat{\rm T}_y^{-1} = \widehat{\rho}_{\rm GE} \ , \tag{174}$$

or, equivalently, the density operator commutes with the generators of translations. We have

$$\begin{aligned}
\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} &= {\rm Tr}\left[\widehat{\rho}_{\rm GE}\, \widehat{T}^{\mu\nu}(x)\right] \\
&= {\rm Tr}\left[\widehat{\rho}_{\rm GE}\, \widehat{\rm T}_{-x}^{-1}\widehat{\rm T}_{-x}\, \widehat{T}^{\mu\nu}(x)\, \widehat{\rm T}_{-x}^{-1}\widehat{\rm T}_{-x}\right] \\
&= {\rm Tr}\left[\left(\widehat{\rm T}_{-x}\widehat{\rho}_{\rm GE}\, \widehat{\rm T}_{-x}^{-1}\right)\left(\widehat{\rm T}_{-x}\, \widehat{T}^{\mu\nu}(x)\, \widehat{\rm T}_{-x}^{-1}\right)\right] \ ,
\end{aligned}$$

where in the last equality we used the cyclic property of the trace. Then, using the transformation properties above, we get

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} = {\rm Tr}\left[\widehat{\rho}_{\rm GE}\, \widehat{T}^{\mu\nu}(0)\right] = \langle \widehat{T}^{\mu\nu}(0)\rangle_{\rm GE} \ .$$

Therefore, in homogeneous global equilibrium, the expectation value has no explicit dependence on the spacetime point x. In local equilibrium, instead, one promotes the constant thermodynamic parameters to slowly varying fields, such as $\beta^\mu = \beta^\mu(x)$ and $\zeta = \zeta(x)$, and the leading-order contribution is obtained by evaluating the global-equilibrium result locally.

Another property we must consider is how the expectation value transforms under a general Lorentz transformation. A Lorentz transformation Λ is represented on the Hilbert space by a corresponding unitary operator $\widehat{\Lambda}$. The stress-energy tensor operator transforms as

$$\widehat{\Lambda}\, \widehat{T}^{\mu\nu}(x)\, \widehat{\Lambda}^{-1} = (\Lambda^{-1})^\mu{}_\lambda (\Lambda^{-1})^\nu{}_\sigma \, \widehat{T}^{\lambda\sigma}(\Lambda x) \ , \tag{175}$$

where $\Lambda x^\alpha \equiv \Lambda^\alpha{}_\beta x^\beta$ is the Lorentz-transformed coordinate vector.

To understand how the global-equilibrium density operator transforms under a Lorentz transformation, we use the fact that a generic function of an operator is defined by its Taylor series. Hence,

$$\begin{aligned}
\widehat{\Lambda}\, \widehat{\rho}_{\rm GE}\, \widehat{\Lambda}^{-1} &= \frac{1}{Z}\widehat{\Lambda}\, {\rm e}^{-\beta\cdot\widehat{P}}\, \widehat{\Lambda}^{-1} \\
&= \frac{1}{Z}\widehat{\Lambda}\left(\widehat{\rm I} - \beta_\lambda \widehat{P}^\lambda + \frac{1}{2}\beta_{\lambda_1}\beta_{\lambda_2}\widehat{P}^{\lambda_1}\widehat{P}^{\lambda_2} - \dots\right)\widehat{\Lambda}^{-1} \\
&= \frac{1}{Z}\left(\widehat{\rm I} - \beta_\lambda \widehat{\Lambda}\,\widehat{P}^\lambda\,\widehat{\Lambda}^{-1} + \frac{1}{2}\beta_{\lambda_1}\beta_{\lambda_2}\widehat{\Lambda}\,\widehat{P}^{\lambda_1}\,\widehat{\Lambda}^{-1}\widehat{\Lambda}\,\widehat{P}^{\lambda_2}\,\widehat{\Lambda}^{-1} - \dots\right) \\
&= \frac{1}{Z}{\rm e}^{-\beta_\lambda \widehat{\Lambda}\widehat{P}^\lambda\widehat{\Lambda}^{-1}} \ .
\end{aligned}$$

Taking into account that $\widehat{P}^\lambda$ is a four-vector operator,

$$\widehat{\Lambda}\,\widehat{P}^\lambda\,\widehat{\Lambda}^{-1} = (\Lambda^{-1})^\lambda{}_\gamma \widehat{P}^\gamma \ ,$$

we get

$$\beta_\lambda \widehat{\Lambda}\,\widehat{P}^\lambda\,\widehat{\Lambda}^{-1} = \beta_\lambda (\Lambda^{-1})^\lambda{}_\gamma \widehat{P}^\gamma = (\Lambda\beta)_\gamma \widehat{P}^\gamma \ .$$

Therefore,

$$\widehat{\Lambda}\,\widehat{\rho}_{\rm GE}(\beta)\,\widehat{\Lambda}^{-1} = \widehat{\rho}_{\rm GE}(\Lambda\beta) \ .$$

Then

$$\begin{aligned}
\langle \widehat{T}^{\mu\nu}(x)\rangle_{{\rm GE},\beta} &= {\rm Tr}\left[\widehat{\rho}_{\rm GE}(\beta)\,\widehat{T}^{\mu\nu}(x)\right] \\
&= {\rm Tr}\left[\widehat{\rho}_{\rm GE}(\beta)\,\widehat{\Lambda}^{-1}\widehat{\Lambda}\,\widehat{T}^{\mu\nu}(x)\,\widehat{\Lambda}^{-1}\widehat{\Lambda}\right] \\
&= {\rm Tr}\left[\left(\widehat{\Lambda}\widehat{\rho}_{\rm GE}(\beta)\widehat{\Lambda}^{-1}\right)\left(\widehat{\Lambda}\,\widehat{T}^{\mu\nu}(x)\,\widehat{\Lambda}^{-1}\right)\right] \\
&= (\Lambda^{-1})^\mu{}_\sigma (\Lambda^{-1})^\nu{}_\gamma\, {\rm Tr}\left[\widehat{\rho}_{\rm GE}(\Lambda\beta)\,\widehat{T}^{\sigma\gamma}(\Lambda x)\right] \ .
\end{aligned}$$

Combining the last line with translation invariance, we can write

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{{\rm GE},\beta} = (\Lambda^{-1})^\mu{}_\sigma (\Lambda^{-1})^\nu{}_\gamma\, \langle \widehat{T}^{\sigma\gamma}(x)\rangle_{{\rm GE},\,\Lambda\beta} \ . \tag{176}$$

Equivalently, this equation states that the expectation value is a rank-two tensor-valued function of the four-vector β^μ.

The above result means that we can compute the components of the expectation value in a preferred reference frame and then obtain the general expression by transforming back. The natural choice is the comoving reference frame, namely the frame where the fluid is at rest:

$$\Lambda\beta^\sigma = \frac{u_0^\sigma}{T} = \frac{1}{T}(1,\mathbf{0})^\sigma \ . \tag{177}$$

In this frame, the equilibrium density operator is invariant under ordinary spatial rotations. Therefore, the expectation value must be rotationally invariant. This implies that, in the comoving frame,

$$\langle \widehat{T}^{00}\rangle_{\rm GE} \neq 0 \ , \qquad \langle \widehat{T}^{0i}\rangle_{\rm GE} = 0 \ , \qquad \langle \widehat{T}^{ij}\rangle_{\rm GE} \propto \delta^{ij} \ . \tag{178}$$

Equivalently, in covariant language, the only available vector on which the expectation value can depend is the four-temperature β^μ. Since $\widehat{T}^{\mu\nu}$ is symmetric, the only possible symmetric rank-two tensors that can be constructed from β^μ and the metric are $\beta^\mu\beta^\nu$ and $g^{\mu\nu}$. Hence,

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} = \mathcal{A}(\beta^2)\beta^\mu\beta^\nu + \mathcal{B}(\beta^2) g^{\mu\nu} \ . \tag{179}$$

Here the scalar coefficients can depend only on the Lorentz scalar $\beta^2 = \beta_\mu\beta^\mu$. Indeed they cannot depend on the coordinates given the homogeneity and being Lorentz scalars they can only depend on the proper Lorentz scalars I can build with the vectors and tensor at my disposal which for the case at hand is only $\beta_\mu\beta_\nu g^{\mu\nu}$. Also, since

$$\beta^\mu = \frac{u^\mu}{T} \ , \qquad \beta^2 = \frac{1}{T^2} \ ,$$

the expression (179) can be equivalently written as:

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} = \widetilde{\mathcal{A}}(\beta^2) u^\mu u^\nu + \mathcal{B}(\beta^2) g^{\mu\nu} \ , \tag{180}$$

where the new coefficients is a simply temperature rescaling of the original one

$$\widetilde{\mathcal{A}}(\beta^2) = \frac{\mathcal{A}(\beta^2)}{T^2} = \mathcal{A}(\beta^2)\beta^2 \ .$$

Now we it is easy to prove that the two coefficients $\mathcal{A}$ and $\mathcal{B}$ are directly related with energy density and pressure and thus the expansion (179) is equivalent to the ideal form given in the hydrodynamic lectures. Starting from (179) and using $\beta^\mu = u^\mu/T$, we obtain:

$$\langle \widehat{T}^{\mu\nu} \rangle_{\rm GE} = \mathcal{A}(\beta^2)\beta^2 u^\mu u^\nu + \mathcal{B}(\beta^2) g^{\mu\nu} \ .$$

We now identify the energy density by projecting twice along the four-velocity:

$$\varepsilon = u_\mu u_\nu \langle \widehat{T}^{\mu\nu} \rangle_{\rm GE} \ .$$

Therefore,

$$\begin{aligned} \varepsilon &= u_\mu u_\nu \left[\mathcal{A}(\beta^2)\beta^\mu\beta^\nu + \mathcal{B}(\beta^2) g^{\mu\nu} \right] \\ &= \mathcal{A}(\beta^2)(u\cdot\beta)^2 + \mathcal{B}(\beta^2) u_\mu u^\mu \\ &= \mathcal{A}(\beta^2)\beta^2 + \mathcal{B}(\beta^2) \ . \end{aligned}$$

The pressure is obtained from the spatial projection:

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \ . \tag{181}$$

Then:

$$p = -\frac{1}{3}\Delta_{\mu\nu}\langle \widehat{T}^{\mu\nu} \rangle_{\rm GE} \ .$$

Using $\Delta_{\mu\nu}u^\nu = 0$ and $\beta^\mu \propto u^\mu$, we have

$$\Delta_{\mu\nu}\beta^\mu\beta^\nu = 0 \ .$$

Moreover,

$$\Delta_{\mu\nu} g^{\mu\nu} = 3 \ .$$

Hence

$$p = -\frac{1}{3}\mathcal{B}(\beta^2)\Delta_{\mu\nu}g^{\mu\nu} = -\mathcal{B}(\beta^2) \ , \tag{182}$$

which finally leads to the identification:

$$\mathcal{B}(\beta^2) = -p \ , \tag{183}$$

and

$$\varepsilon = \mathcal{A}(\beta^2)\beta^2 - p \ . \tag{184}$$

The above can be expressed in the following equivalent way

$$\mathcal{A}(\beta^2)\beta^2 = \varepsilon + p \ .$$

so that substituting these identifications back into the covariant decomposition, we finally obtain

$$\langle \widehat{T}^{\mu\nu} \rangle_{\rm GE} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu} \ . \tag{185}$$

This is precisely the ideal-fluid form of the stress-energy tensor.

E. Exercise 5

In this exercise we analyze the properties of the dissipative corrections for the stress-energy tensor. Considering only the four-temperature contribution we have:

$$\langle \widehat{T}^{\mu\nu}(x) \rangle \simeq \langle \widehat{T}^{\mu\nu}(x) \rangle_{\rm GE} + \int_\Omega {\rm d}^4 y \int_0^1 {\rm d}z \, \left\langle \widehat{T}^{\mu\nu}(x),\, {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}} \, \widehat{T}^{\rho\sigma}(y) \, {\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}} \right\rangle_{c,{\rm GE}} \partial_\rho \beta_\sigma(y) \ . \tag{186}$$

The correction depends on the myscroscopic correlator:

$$K^{\mu\nu,\rho\sigma}(y,x) \equiv \int_0^1 \mathrm{d}z \, \left\langle \widehat{T}^{\mu\nu}(x),\, \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \right\rangle_{c,\mathrm{GE}} \, , \tag{187}$$

so that the (186) can be equivalently written as:

$$\langle \widehat{T}^{\mu\nu}(x) \rangle \simeq \langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} + \int_\Omega \mathrm{d}^4 y \, K^{\mu\nu,\rho\sigma}(y,x)\, \partial_\rho \beta_\sigma(y) \, . \tag{188}$$

Now we want to prove that the correlator (187) defining the leading order correction to the stress-energy tensor actually depends on $y - x$. This is a direct consequence of the fact that the stress tensor is a local operator and that in linear response theory the expectation value is computed on the homogeneous equilibrium state.

To show this lets consider the definition of expectation value:

$$\begin{aligned} \left\langle \widehat{T}^{\mu\nu}(x),\, \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \right\rangle_{c,\mathrm{GE}} &= \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}\, \widehat{T}^{\mu\nu}(x) \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \right] \\ &\quad - \langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} \langle \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \rangle_{\mathrm{GE}} \, . \end{aligned}$$

The second term, due to the ciclic property of the trace simply reduces to the disconnected part of $\widehat{T}(x)\widehat{T}(y)$ and thus at homogeneous global equilibrium is independent on x has we proved in the previous exercise. We can thus concentrate on the first part. Using the transformation law of the stress tensor under translation:

$$\widehat{\mathrm{T}}_y\, \widehat{T}^{\mu\nu}(x)\, \widehat{\mathrm{T}}_y^{-1} = \widehat{T}^{\mu\nu}(x+y) \, , \qquad \widehat{\mathrm{T}}_y = \mathrm{e}^{\mathrm{i}y\cdot\widehat{P}} \, , \tag{189}$$

we have:

$$\begin{aligned} \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}\, \widehat{T}^{\mu\nu}(x) \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \right] &= \frac{1}{Z} \left[\mathrm{e}^{-\beta\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(x) \mathrm{e}^{z\beta\cdot\widehat{P}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] \\ &= \frac{1}{Z} \left[\mathrm{e}^{-\beta\cdot\widehat{P}}\, \mathrm{e}^{\mathrm{i}x\cdot\widehat{P}} \widehat{T}^{\mu\nu}(0) \mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}} \mathrm{e}^{z\beta\cdot\widehat{P}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] \\ &= \frac{1}{Z} \left[\mathrm{e}^{\mathrm{i}x\cdot\widehat{P}} \mathrm{e}^{-\beta\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(0) \mathrm{e}^{z\beta\cdot\widehat{P}}\, \mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}} \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] \, , \end{aligned}$$

where we used that the various exponents of $\widehat{P}$ commute with each other. Now using the ciclic property of the trace we can move the factor $\exp(\mathrm{i}x\cdot\widehat{P})$ on the right and commuting again we get:

$$\begin{aligned} \frac{1}{Z} \left[\mathrm{e}^{\mathrm{i}x\cdot\widehat{P}} \mathrm{e}^{-\beta\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(0) \mathrm{e}^{z\beta\cdot\widehat{P}}\, \mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}} \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] &= \frac{1}{Z} \left[\mathrm{e}^{-\beta\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(0) \mathrm{e}^{z\beta\cdot\widehat{P}}\, \mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}} \widehat{T}^{\rho\sigma}(y) \mathrm{e}^{\mathrm{i}x\cdot\widehat{P}}\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] \\ &= \frac{1}{Z} \left[\mathrm{e}^{-\beta\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(0) \mathrm{e}^{z\beta\cdot\widehat{P}}\, \widehat{T}^{\rho\sigma}(y-x)\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] \, , \end{aligned}$$

that is:

$$\mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}\, \widehat{T}^{\mu\nu}(x) \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\, \widehat{T}^{\rho\sigma}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \right] = \frac{1}{Z} \left[\mathrm{e}^{-\beta\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(0) \mathrm{e}^{z\beta\cdot\widehat{P}}\, \widehat{T}^{\rho\sigma}(y-x)\, \mathrm{e}^{-z\beta\cdot\widehat{P}} \right] \, ,$$

which together with the constancy of the disconnected part implies:

$$K^{\mu\nu,\rho\sigma}(y,x) = K^{\mu\nu,\rho\sigma}(y-x) \, . \tag{190}$$

Note that replacing the first $\widehat{T}^{\mu\nu}(x)$ with a generic local operator the corresponding correlator would still only depend on $y - x$.