Lecture notes BIP Timisoara 2026

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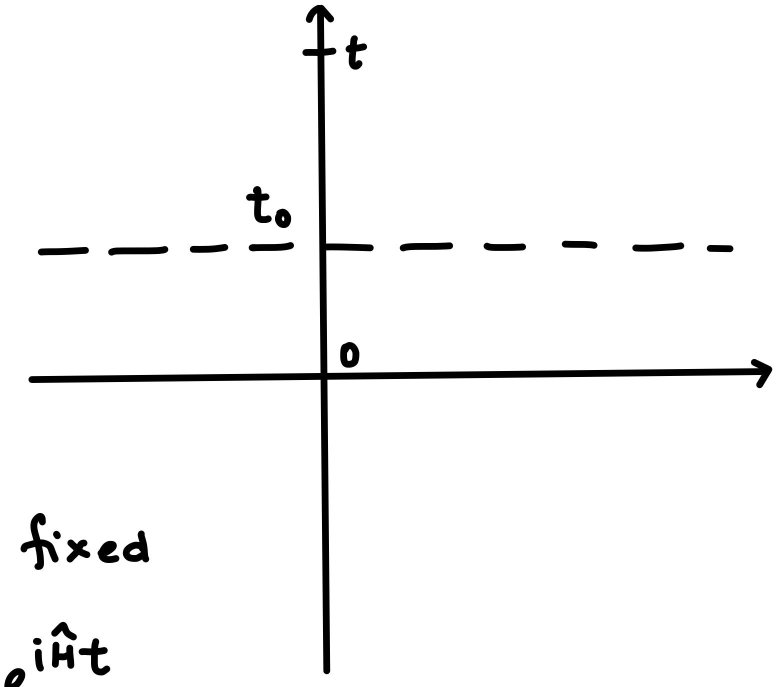
Dynamics

$\hat{\rho} = ?$

$$T^{\mu\nu}(\underline{x},t) = \mathrm{Tr}\left(\hat{\rho}(0)\,\hat{T}^{\mu\nu}(\underline{x},t)\right)$$



Heisenberg $\quad \hat{T}^{\mu\nu}(\underline{x},t) = e^{i\hat{H}t}\,\hat{T}^{\mu\nu}(\underline{x},0)\,e^{-i\hat{H}t} \quad \hat{\rho}$ fixed

Schrödinger $\quad \hat{T}^{\mu\nu}(\underline{x},0)$ fixed $\quad \hat{\rho}(t) = e^{-i\hat{H}t}\hat{\rho}(0)e^{i\hat{H}t}$

Suppose that a good approximation can be found of

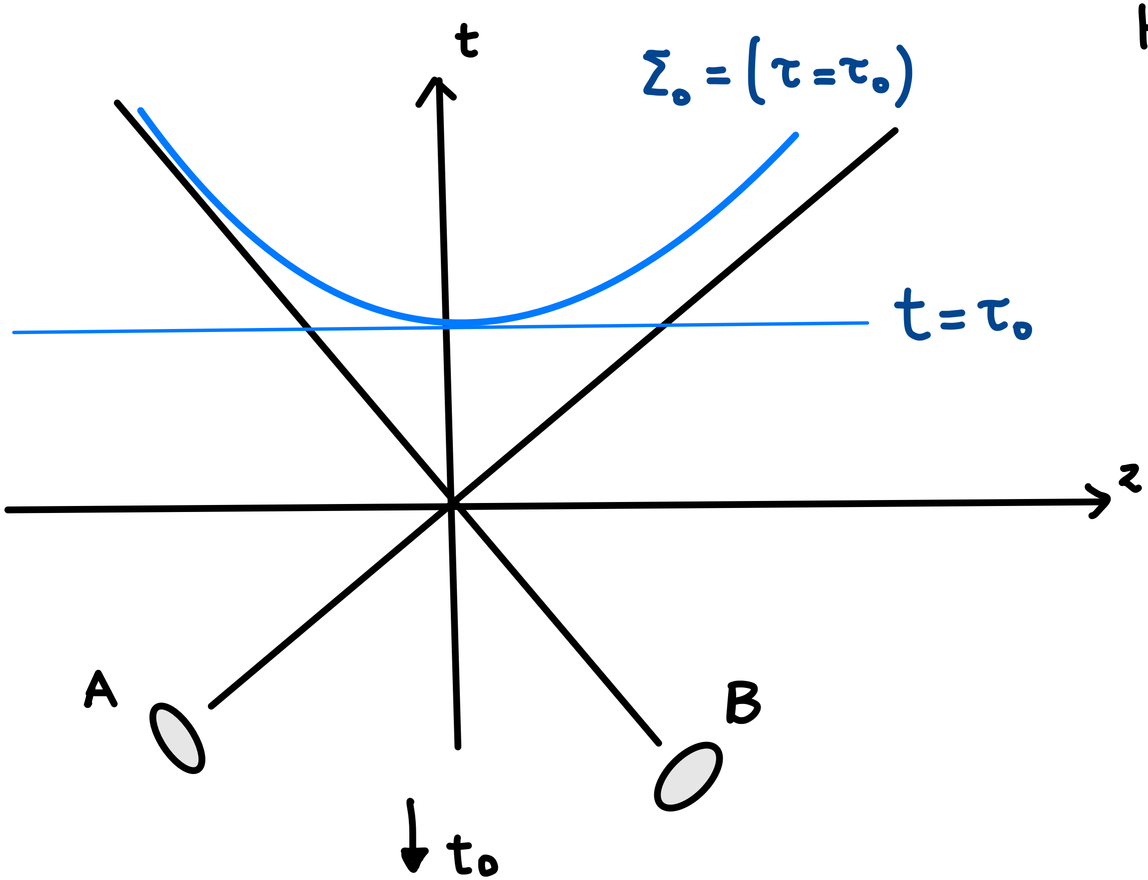
$\hat{\rho}(t_0) = e^{-i\hat{H}t_0}\hat{\rho}(0)e^{i\hat{H}t_0}$ at a suitable time t_0

$\rightarrow$ evolve $\hat{T}^{\mu\nu}$ or any other fields from t_0

Relativistic heavy ion collisions

$$\hat{\rho}_{true} = |P_{1_A} P_{2_B}\rangle\langle P_{1_A} P_{2_B}|$$

$$\hat{\rho}_S(t=\tau_0)$$
$$= e^{-i\hat{H}(\tau_0-t_0)}\,\hat{\rho}_{true}\,e^{i\hat{H}(\tau_0-t_0)}$$



$$\hat{H} = \int d^3x\,\hat{T}^{00} = \int_{\Sigma_0} d\Sigma_\mu \hat{T}^{\mu\nu}\hat{t}_\nu$$

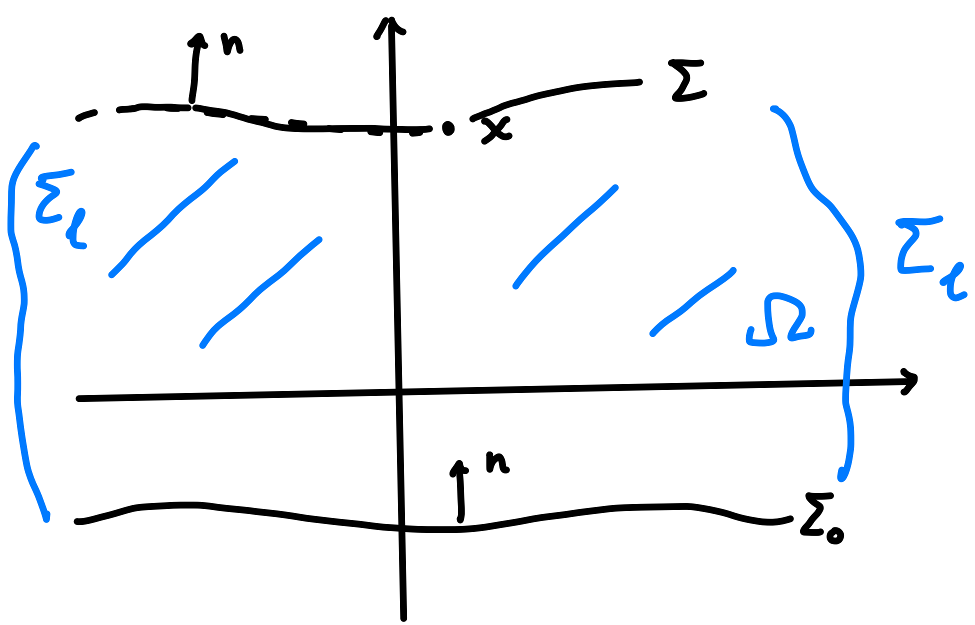
Gauss theorem

Assumption

$$\hat{\rho}(\tau_0) = \hat{\rho}_{LE}(\Sigma_0)$$

so $\hat{\rho} = \hat{\rho}_{LE}(\Sigma_0)$

GAUSS THEOREM



$$-\int_{\Sigma_0} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu + \int_{\Sigma} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu =$$

$$= \int d\Omega\, \nabla_\mu (\hat{T}^{\mu\nu}\beta_\nu)$$

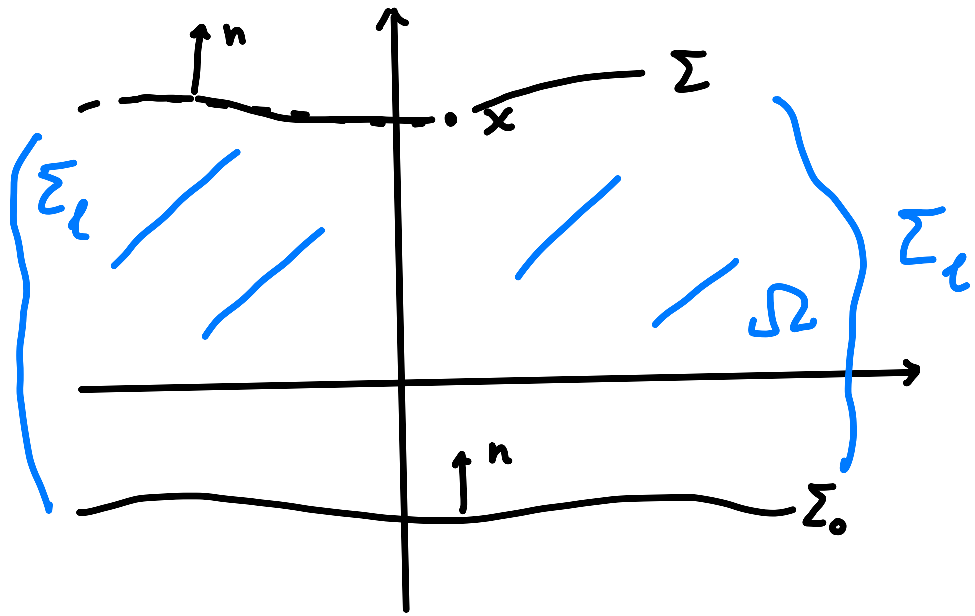
$$\text{if} \quad \int_{\Sigma_\ell} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu = 0$$

$$\boxed{-\int_{\Sigma_0} d\Sigma\, \hat{T}^{\mu\nu}\beta_\nu = -\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu + \int d\Omega\, \hat{T}^{\mu\nu}\nabla_\mu\beta_\nu}$$

$$\Rightarrow \quad \hat{\rho}_{LE}(\Sigma_0) = \frac{1}{Z}\exp\left[-\int d\Sigma_\mu\left(\hat{T}^{\mu\nu}\beta_\nu - \zeta\hat{j}^\mu\right) + \int d\Omega\left(\hat{T}^{\mu\nu}\nabla_\mu\beta_\nu - \nabla_\mu\zeta\cdot\hat{j}^\mu\right]\right.$$

$$T^{\mu\nu}(x) = \mathrm{Tr}\left(\hat{\rho}\, \hat{T}^{\mu\nu}(x)\right) = \mathrm{Tr}\left(\hat{\rho}_{LE}(\Sigma_0)\, \hat{T}^{\mu\nu}(x)\right)$$

$$T^{\mu\nu}_{LE}(x) = \mathrm{Tr}\left(\frac{e^{-\int_\Sigma d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu}}{Z} \hat{T}^{\mu\nu}(x)\right)$$



Remember

$$n_\mu T^{\mu\nu} = n_\mu T^{\mu\nu}_{LE} \quad \text{on } \Sigma$$

Linear Response Theory

$$\hat{\rho} = \frac{1}{Z} e^{\hat{A}+\hat{B}} \qquad Z = \mathrm{Tr}\left(e^{\hat{A}+\hat{B}}\right)$$

Kubo identity

$$e^{\hat{A}+\hat{B}} = \left[I + \int_0^1 dz\, e^{z(\hat{A}+\hat{B})} \hat{B} e^{-z\hat{A}} \right] e^{\hat{A}}$$ can be iterated

$$\Rightarrow e^{\hat{A}+\hat{B}} \simeq \left[I + \int_0^1 dz\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \right] e^{\hat{A}}$$ linear response

$$\hat{\rho} \simeq \hat{\rho}_A \left(1 - \langle \hat{B} \rangle_A\right) + \int_0^1 dz\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \hat{\rho}_A \qquad \hat{\rho}_A = \frac{e^{\hat{A}}}{\mathrm{Tr}(e^{\hat{A}})}$$

Project: expand $\frac{1}{Z} e^{-b\cdot\hat{P} + \frac{1}{2}\varpi:\hat{J}}$ at the linear order in ϖ taking into account the commutation relations $[\hat{P}^\mu, \hat{J}^{\lambda\nu}]$

Example:

$$\hat{A} = -\int_{\Sigma} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu \qquad \hat{\rho} \cong \hat{\rho}_{LE}\left(1 - \langle \hat{B} \rangle_{LE}\right) + \int_0^1 dz\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \hat{\rho}_{LE}$$

$$\hat{B} = \int d^4y\, \hat{T}^{\mu\nu}\partial_\mu\beta_\nu - \partial_\mu\zeta\, \hat{j}^\mu \qquad \langle \hat{T}^{\mu\nu}(x) \rangle = \mathrm{Tr}\left(\hat{\rho}\, \hat{T}^{\mu\nu}(x)\right)$$

$$\langle \hat{T}^{\mu\nu}(x) \rangle \cong \langle \hat{T}^{\mu\nu}(x) \rangle_{LE} + \int_0^1 dz \langle \hat{T}^{\mu\nu}(x)\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \rangle_{LE} - \langle \hat{T}^{\mu\nu}(x) \rangle_{LE} \langle \hat{B} \rangle_{LE}$$

$$= \langle \hat{T}^{\mu\nu}(x) \rangle_{LE} + \int_0^1 dz \langle \hat{T}^{\mu\nu}(x),\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \rangle_{LE,c} \qquad \langle \hat{X}, \hat{Y} \rangle_c = \langle \hat{X}\hat{Y} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle$$

$$= \langle \hat{T}^{\mu\nu}(x) \rangle_{LE} + \int_0^1 dz \int d^4y \langle \hat{T}^{\mu\nu}(x),\, e^{z\hat{A}} \hat{T}^{\rho\sigma}(y) e^{-z\hat{A}} \rangle_{LE,c}\, \partial_\rho\beta_\sigma(y) + \ldots\ldots$$

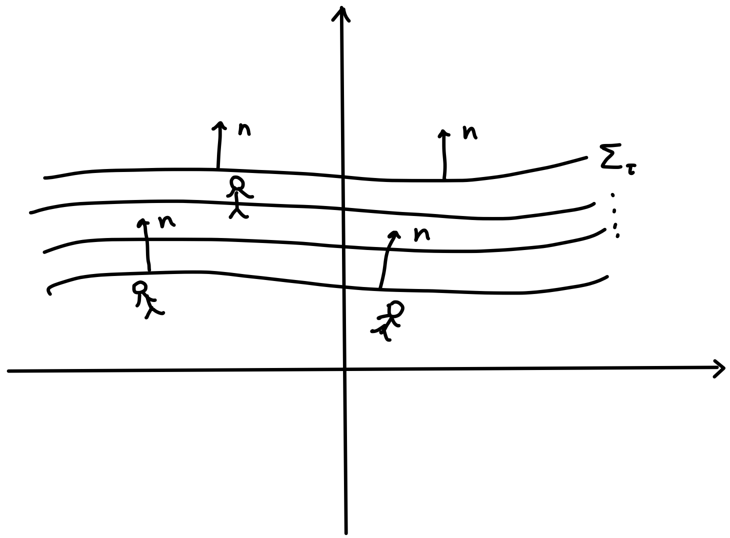
Hydro approx. $\partial_\rho\beta_\sigma(y) \simeq \partial_\rho\beta_\sigma(x)$

$$\Rightarrow \quad \langle \hat{T}^{\mu\nu}(x) \rangle_{LE} + \partial_\rho\beta_\sigma(x) \int_0^1 dz \int d^4y \langle \hat{T}^{\mu\nu}(x),\, e^{z\hat{A}} \hat{T}^{\rho\sigma}(y) e^{-z\hat{A}} \rangle_{LE,c} + \ldots\ldots$$

This is a constitutive equation

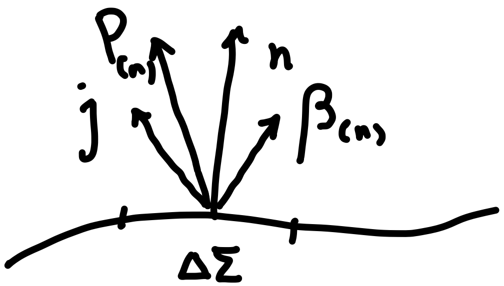
$$T^{\mu\nu} \cong \langle \hat{T}^{\mu\nu}(x) \rangle_{LE} + \partial_\rho \beta_\sigma(x) \underbrace{\int_0^1 dz \int d^4y \, \langle \hat{T}^{\mu\nu}(x), e^{z\hat{A}} \hat{T}^{\rho\sigma}(y) e^{-z\hat{A}} \rangle_{LE,c}}_{\text{encodes transport coefficients}} + \ldots$$

The separation between LE and dissipative corrections depend on the chosen foliation



Relation with RH

frame in class. RH $\longleftrightarrow$ foliation in QS framework



$$P_{(n)} = T^{\mu\nu} n_\mu \Delta\Sigma$$

$$\beta_{(n)} = \frac{\partial S_{(n)}}{\partial P_{(n)}} \qquad S_{(n)} = s^\mu n_\mu \Delta\Sigma$$

n is arbitrary, but it can be

① $n \parallel P_{(n)}$ ~ Landau frame $\Rightarrow T^{\mu\nu} n_\mu \Delta\Sigma = \lambda n^\nu$ $n \equiv u_L$ eigenvector of $T^{\mu\nu}$

② $n \parallel j$ ~ Eckart frame $n \propto j$

③ $\beta \parallel n \longrightarrow \beta$ or thermometric frame $u = \beta / \sqrt{\beta^2}$

They all suffer the vorticity problem

$$\epsilon^{\mu\rho\sigma\tau}\partial_\rho n_\sigma n_\tau = 0 \rightarrow \text{for } \Sigma \text{ to exist}$$

Hydro is a dynamical problem

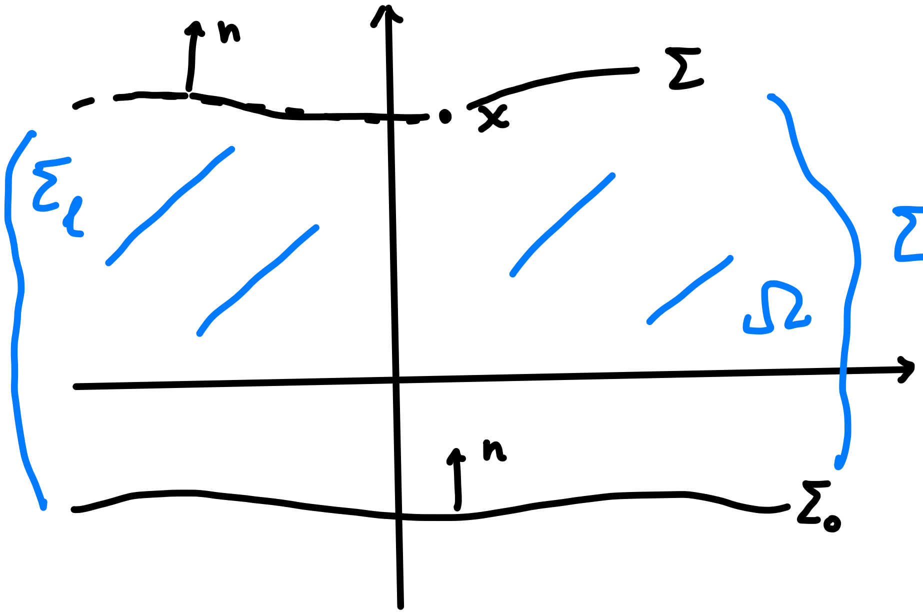
Evolve $\partial_\mu \mathrm{Tr}(\hat{\rho}\hat{T}^{\mu\nu}(x)) = 0$ from conditions on a specific Σ_0

However, the separation between LE and DISS requires a foliation

KUBO FORMULAE DERIVATION

From linear response

$$T^{\mu\nu}(x) = \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \int_0^1 dz \int_\Omega d^4y \, \langle \hat{T}^{\mu\nu}(x), e^{z\hat{A}}\hat{T}^{\rho\sigma}(y)e^{-z\hat{A}}\rangle_{LE,c}\, \partial_\rho \beta_\sigma(y) + \ldots\ldots$$



$\downarrow$

$\simeq$ GE in $\beta(x), \zeta(x)$

$$\hat{A} \simeq -\beta(x)\cdot\hat{P} + \zeta\hat{Q}$$

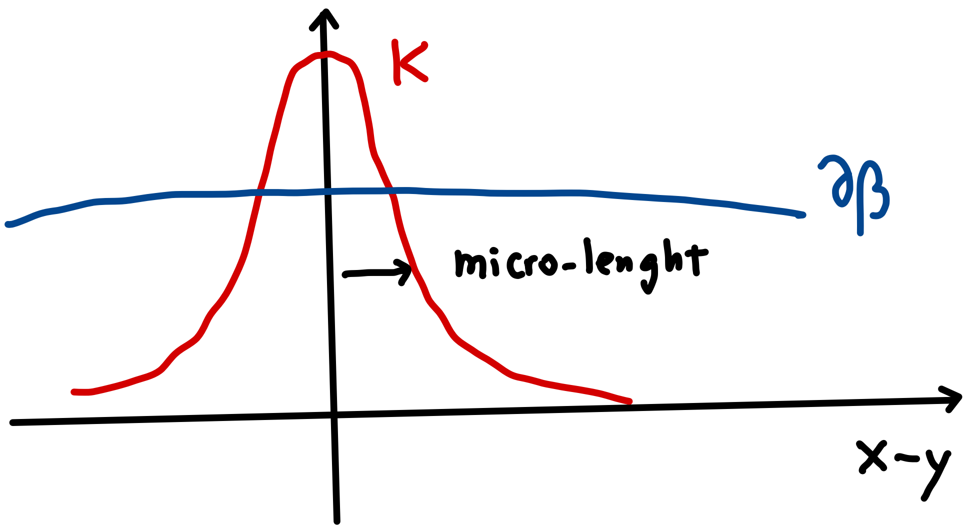
$$T^{\mu\nu}(x) \simeq \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \int_0^1 dz \int_\Omega d^4y \, \langle \hat{T}^{\mu\nu}(x), e^{z\hat{A}}\hat{T}^{\rho\sigma}(y)e^{-z\hat{A}}\rangle_{GE,c}\, \partial_\rho \beta_\sigma(y) + \ldots\ldots$$

$$T^{\mu\nu}(x) \simeq \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \int_\Omega d^4y \underbrace{\int_0^1 dz \langle \hat{T}^{\mu\nu}(x), e^{z\hat{A}}\hat{T}^{\rho\sigma}(y)e^{-z\hat{A}}\rangle_{GE,c}}_{K^{\mu\nu\rho\sigma}(x-y)} \partial_\rho \beta_\sigma(y) + \ldots\ldots$$

Problem: prove that K is a function of $x-y$ (T)

Usually:

$$T^{\mu\nu}(x) \simeq \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \partial_\rho\beta_\sigma(x) \cdot \underbrace{\int_\Omega d^4y\, K^{\mu\nu\rho\sigma}(x-y)}_{\text{encodes } \eta,\, \zeta \text{ etc.}}$$



Alternatively:

$$\int d^4y\, K^{\mu\nu\rho\sigma}(x-y)\,\partial_\rho\beta_\sigma(y) = \int d^4y\, K^{\mu\nu\rho\sigma}(x-y) \int d^4k\, \widetilde{\partial_\rho\beta_\sigma}(k)\, e^{ik\cdot y}$$

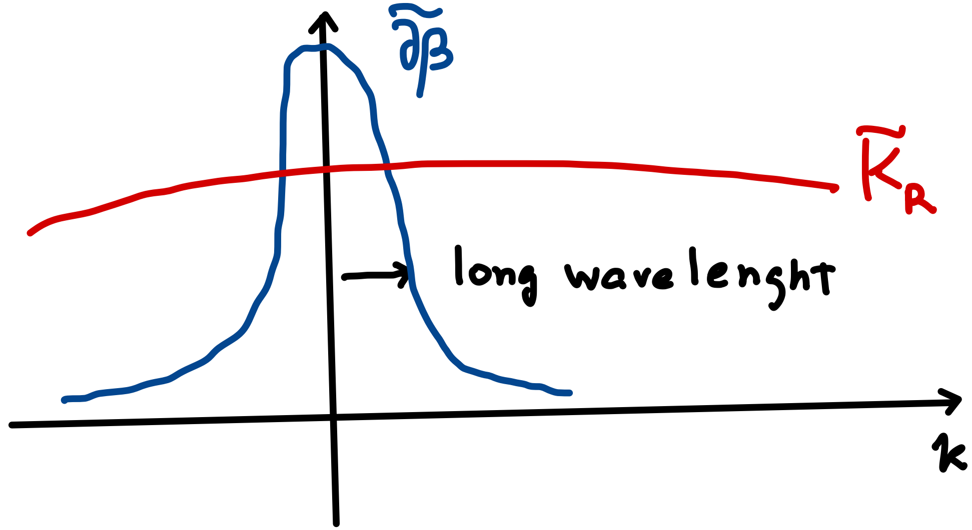
$$= \int d^4k \int d^4y\, K^{\mu\nu\rho\sigma}(x-y)\, e^{-ik\cdot(x-y)}\, \widetilde{\partial_\rho\beta_\sigma}(k)\, e^{ik\cdot x}$$

Geometric assumption : $\Omega = \mathbb{R}^3 \times (-\infty, t]$

$$= \int d^4k \int d^4y\, \theta(x^0-y^0)\, K^{\mu\nu\rho\sigma}(x-y)\, e^{-ik\cdot(x-y)}\, \widetilde{\partial_\rho\beta_\sigma}(k)\, e^{ik\cdot x}$$

$$= \int d^4k\, \widetilde{K_R^{\mu\nu\rho\sigma}}(k)\, \widetilde{\partial_\rho\beta_\sigma}(k)\, e^{ik\cdot x}$$

Just the convolution theorem!



$$\int d^4k\, \widetilde{K_R^{\mu\nu\rho\sigma}(k)}\, \widetilde{\partial_\rho\beta_\sigma(k)}\, e^{ik\cdot x} \simeq \widetilde{K_R^{\mu\nu\rho\sigma}(0)} \int d^4k\, \widetilde{\partial_\rho\beta_\sigma(k)}\, e^{ik\cdot x}$$

$$= \widetilde{K_R^{\mu\nu\rho\sigma}(0)} \cdot \partial_\rho\beta_\sigma(x)$$

Compare with previous : $\partial_\rho\beta_\sigma(x) \cdot \int d^4y\, \Theta(x^0 - y^0) K^{\mu\nu\rho\sigma}(x-y)$

$$\widetilde{K_R^{\mu\nu\rho\sigma}(0)} = \lim_{k\to 0} \int_{-\infty}^{x^0} d^4y \int_0^1 dz \left\langle \hat{T}^{\mu\nu}(x), e^{z\hat{A}}\hat{T}^{\rho\sigma}(y)e^{-z\hat{A}} \right\rangle_{GE,c} e^{-ik\cdot(x-y)}$$

Same formula if the limit $k\to 0$ can be moved inside

One can play around with integrations and finally come to

$$\widetilde{K_R^{\mu\nu\rho\sigma}(0)} = -\frac{d}{dk^0}\lim_{\underline{k}\to 0}\int_{-\infty}^{x^0} d^4y \left\langle \left[\hat{T}^{\mu\nu}(x), \hat{T}^{\rho\sigma}(y) \right] \right\rangle_{GE} e^{-ik\cdot(x-y)}$$

$$+ \lim_{k\to 0} \int_{-\infty}^{x^0} d^4y \int_0^1 dz \left\langle \hat{T}^{\mu\nu}(x), e^{z\hat{A}} \hat{T}^{\rho\sigma}(-\infty, \underline{y}) e^{-z\hat{A}} \right\rangle_{GE,c} e^{-ik\cdot(x-y)}$$

Missing steps (T)