Tutoring 29/05

Daniele Roselli[1]

[1] *Università degli studi di Firenze and INFN Sezione di Firenze, Via G. Sansone 1, I-50019 Sesto Fiorentino (Florence), Italy*

I. EXERCISE 3

Let us start with the rotating case. We choose the rotation axis to be the z axis. The corresponding parameters are

$$b_\mu = \frac{1}{T_0}\left(1, \mathbf{0}\right) \ , \qquad \varpi_{\mu\nu} = \frac{\omega}{T_0}\left(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu}\right) \ .$$

Here T_0 is not the proper temperature at a generic point of the rotating system. It is the temperature measured on the rotation axis, where the local velocity of the rotating medium vanishes.

Substituting this choice into the global-equilibrium statistical operator gives

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\frac{\widehat{H}}{T_0} + \frac{\omega}{T_0}\widehat{J}_z\right) \ ,$$

where $\widehat{J}_z = \widehat{J}^{12}$ is the generator of rotations around the z axis. This is the relativistic version of the density operator for a system in rigid rotation: the Hamiltonian is shifted by the angular-momentum generator, weighted by the angular velocity.

For a generic rotation axis, one can write

$$b_\mu = \frac{1}{T_0}\left(1, \mathbf{0}\right) \ , \qquad \varpi \cdot x = \left(0, \boldsymbol{\omega} \times \mathbf{x}\right) \ ,$$

and the replacement

$$\omega \widehat{J}_z \mapsto \boldsymbol{\omega} \cdot \widehat{\mathbf{J}}$$

has to be made.

The corresponding four-temperature field is

$$\beta^\mu = \frac{1}{T_0}\left(1, \boldsymbol{\omega} \times \mathbf{x}\right) \ . \tag{1}$$

For rotation around the z axis,

$$\boldsymbol{\omega} = \omega \hat{\mathbf{z}} \ ,$$

so that

$$\boldsymbol{\omega} \times \mathbf{x} = (-\omega y, \omega x, 0)$$

up to the conventional orientation of the rotation. The important point is that the spatial part of β^μ is tangent to circles centered around the rotation axis.

The vector (1) is not timelike everywhere. Its norm is

$$\beta^2 = \frac{1}{T_0^2}\left(1 - |\boldsymbol{\omega} \times \mathbf{x}|^2\right) \ .$$

For rotation around the z axis,

$$|\boldsymbol{\omega} \times \mathbf{x}|^2 = \omega^2 r^2 \ , \qquad r = \sqrt{x^2 + y^2} \ ,$$

and therefore

$$\beta^2 = \frac{1}{T_0^2}\left(1 - \omega^2 r^2\right) \ .$$

The four-temperature is timelike only if

$$1-\omega^2 r^2 > 0 \ ,$$

namely

$$r < \frac{1}{|\omega|} \ .$$

The surface

$$r = \frac{1}{|\omega|}$$

is called the light cylinder. At this radius, the tangential velocity of a rigidly rotating fluid element would reach the speed of light. Therefore, a global equilibrium state with rigid rotation cannot be extended beyond the light cylinder.

The proper temperature is obtained from the norm of the four-temperature:

$$T(x) = \frac{1}{\sqrt{\beta^2(x)}} \ .$$

For the rotating configuration, this gives

$$T(r) = \frac{T_0}{\sqrt{1-\omega^2 r^2}} \ .$$

Thus the proper temperature measured by a comoving observer increases with the distance from the rotation axis. This does not mean that the temperature changes in time for a given fluid element. A fluid element at fixed radius r follows a circular trajectory, and along this trajectory the value of r is constant. Therefore, the proper temperature measured by that fluid element is constant along its motion.

Let us make this geometrical statement more explicit by studying the field lines of β^μ. They are defined by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \beta^\mu(x) \ .$$

Using (1), we get

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{T_0} \ , \qquad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\lambda} = \frac{\boldsymbol{\omega}\times\mathbf{x}}{T_0} \ .$$

Dividing the spatial equation by the time equation gives

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \boldsymbol{\omega}\times\mathbf{x} \ .$$

For rotation around the z axis, this becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\omega y \ , \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \omega x \ , \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0 \ .$$

The solutions are circular trajectories:

$$x(t) = r\cos(\omega t + \phi_0) \ , \qquad y(t) = r\sin(\omega t+\phi_0) \ , \qquad z(t) = z_0 \ .$$

Therefore, the field lines of β^μ are helices in spacetime: the spatial motion is circular, while the time coordinate increases monotonically. These are the worldlines of the comoving fluid elements in rigid rotation.

This is the correct physical interpretation of rotating global equilibrium. The state is not spatially homogeneous, because the proper temperature depends on r. Nevertheless, it is in global equilibrium because the system is stationary along the flow generated by β^μ. Each comoving observer measures a constant temperature along its own worldline, but observers at different radii measure different proper temperatures.

The second basic equilibrium configuration is the accelerated one. This configuration is purely relativistic and has no direct non-relativistic analogue. It is associated with Lorentz boosts rather than with ordinary spatial rotations.

We choose the acceleration to be along the z direction and take

$$b_\mu = \frac{1}{T_0}(1, \mathbf{0}) \,, \qquad \varpi_{\mu\nu} = \frac{a}{T_0}\left(g_{0\mu}g_{3\nu} - g_{0\nu}g_{3\mu}\right) \,.$$

The contraction of the thermal vorticity with the Lorentz generators is proportional to the generator of boosts along the z direction. The statistical operator then reads

$$\widehat{\rho}_{\rm GE} = \frac{1}{Z}\exp\left(-\frac{\widehat{H}}{T_0} + \frac{a}{T_0}\widehat{K}_z\right) \,.$$

This operator is peculiar because it involves the boost generator. The boost generator has an explicit time dependence. With the convention used here, it can be written as

$$\widehat{K}_z = \widehat{J}_{30} = t\widehat{P}_z - \int {\rm d}^3x \, z \, \widehat{T}^{00} \,.$$

Nevertheless, the statistical operator is time independent, as required by global equilibrium. The explicit time dependence of $\widehat{K}_z$ is compensated by its non-vanishing commutator with the Hamiltonian. With the corresponding convention for the time evolution of operators, one has

$$\frac{{\rm d}\widehat{K}_z}{{\rm d}t} = \frac{\partial \widehat{K}_z}{\partial t} - {\rm i}\left[\widehat{H}, \widehat{K}_z\right] = \widehat{P}_z - \widehat{P}_z = 0 \,.$$

Therefore, the boost generator is a conserved generator, and the statistical operator is indeed independent of time.

It is important to understand the physical meaning of the parameter a appearing in the statistical operator. The notation may suggest that the whole fluid is accelerating with proper acceleration a, but this is not correct. The parameter a fixes the normalization of the boost Killing field and identifies a special reference hyperbola, namely the one whose proper acceleration is a. However, a spatially extended accelerated fluid in global equilibrium cannot have the same non-zero proper acceleration at every point. Different fluid elements follow different hyperbolic trajectories, and therefore they have different proper accelerations.

With the above choice of parameters, the four-temperature field can be written in a transparent way by introducing

$$z' = z + \frac{1}{a} \,.$$

Then

$$\beta^\mu = \frac{1}{T_0}\left(1 + az, 0, 0, at\right) = \frac{a}{T_0}\left(z', 0, 0, t\right) \,. \qquad (2)$$

Notice that here we are writing the contravariant components of β^μ. With the metric convention

$$g^{\mu\nu} = {\rm diag}(1, -1, -1, -1) \,,$$

the covariant components are

$$\beta_\mu = \frac{a}{T_0}\left(z', 0, 0, -t\right) \,.$$

This distinction is important because the sign of the spatial covariant component changes.

The vector field (2) is proportional to the boost Killing vector

$$z'\partial_t + t\partial_z \,.$$

Indeed,

$$\beta = \beta^\mu \partial_\mu = \frac{a}{T_0}\left(z'\partial_t + t\partial_z\right) \,.$$

Thus the accelerated equilibrium state is stationary with respect to a boost flow, not with respect to ordinary inertial time translations. This is the essential difference between this configuration and homogeneous equilibrium.

The norm of the four-temperature is

$$\beta^2 = \frac{a^2}{T_0^2}\left(z'^2 - t^2\right) \ .$$

Therefore, the proper temperature field is

$$T(t,z) = \frac{1}{\sqrt{\beta^2}} = \frac{T_0}{a\sqrt{z'^2 - t^2}} \ .$$

The four-temperature is timelike only in the region

$$z'^2 - t^2 > 0 \ .$$

This region consists of two Rindler wedges, shifted by the replacement $z \mapsto z' = z + 1/a$. In the right wedge one has

$$z' > |t| \ .$$

The boundaries

$$z' = \pm t$$

are null surfaces. On these surfaces $\beta^2 = 0$, so the Killing vector becomes lightlike. Therefore, the accelerated equilibrium state is physically meaningful only inside the wedge where β^μ is timelike and future-directed.

Let us now study the field lines of (2). They are defined by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \beta^\mu(x) \ .$$

For the accelerated field this gives

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{a}{T_0} z' \ , \qquad \frac{\mathrm{d}z'}{\mathrm{d}\lambda} = \frac{a}{T_0} t \ ,$$

where $z' = z + 1/a$, so $\mathrm{d}z' = \mathrm{d}z$. Dividing the two equations, we find

$$\frac{\mathrm{d}z'}{\mathrm{d}t} = \frac{t}{z'} \ .$$

Hence

$$z' \,\mathrm{d}z' = t \,\mathrm{d}t \ .$$

Integrating, we obtain

$$z'^2 - t^2 = \rho^2 \ ,$$

where ρ is a positive constant labeling the different field lines. Therefore, the field lines of β^μ are hyperbolae in the (t,z) plane.

These hyperbolae are precisely the worldlines of uniformly accelerated observers. A convenient parametrization is

$$t = \rho \sinh \eta \ , \qquad z' = \rho \cosh \eta \ .$$

Along each trajectory,

$$z'^2 - t^2 = \rho^2 = \text{const.}$$

Therefore, the proper temperature measured by the comoving observer on that trajectory is

$$T = \frac{T_0}{a\rho} \ .$$

This is constant along the observer's worldline. Thus, although the field $T(t,z)$ depends on the Minkowski coordinates t and z, a comoving observer does not measure a temperature that changes with time. The apparent time dependence is due to the fact that Minkowski time is not the natural time coordinate adapted to the accelerated equilibrium flow.

The parameter ρ also determines the proper acceleration of the corresponding worldline. The hyperbola

$$z'^2 - t^2 = \rho^2$$

describes a uniformly accelerated observer with proper acceleration

$$\alpha(\rho) = \frac{1}{\rho} \ .$$

Therefore, along a given accelerated trajectory,

$$T(\rho) = \frac{T_0}{a\rho} = \frac{T_0}{a}\alpha(\rho) \ .$$

This formula clarifies the role of a. The parameter a is not the proper acceleration of every fluid element. Rather, it selects the particular hyperbola

$$\rho = \frac{1}{a} \ .$$

On this hyperbola,

$$\alpha = \frac{1}{\rho} = a \ ,$$

and the proper temperature is

$$T = \frac{T_0}{a(1/a)} = T_0 \ .$$

Thus T_0 is the proper temperature measured by the comoving observer whose proper acceleration is a. Other comoving observers, corresponding to other values of ρ, measure different constant temperatures.

This point is a common source of confusion. The statistical operator does not describe a macroscopic fluid whose every element has the same proper acceleration a. Such a configuration would not correspond to a spatially extended equilibrium fluid. Indeed, a fixed value of the proper acceleration,

$$\alpha = \text{const.} \neq 0 \ ,$$

corresponds to a single hyperbola,

$$z'^2 - t^2 = \frac{1}{\alpha^2} \ .$$

A single hyperbola is one worldline, not a macroscopic fluid filling a spatial region. A spatially extended accelerated fluid must instead be described by a congruence of hyperbolae, labeled by different values of ρ. Since the proper acceleration is $\alpha(\rho) = 1/\rho$, different fluid elements necessarily have different proper accelerations.

This is also the relativistic condition for maintaining a rigid accelerated configuration. If all parts of an extended body had the same non-zero proper acceleration, their distances in the instantaneous comoving frame would not remain constant. In order for an accelerated body or fluid to remain in equilibrium without stretching or compression, different parts must accelerate differently. The boost Killing flow implements exactly this: the fluid elements follow different hyperbolae, and the proper acceleration varies from one hyperbola to another.

We can now introduce coordinates adapted to this accelerated motion. The natural coordinates are Rindler coordinates. In the right Rindler wedge, we define

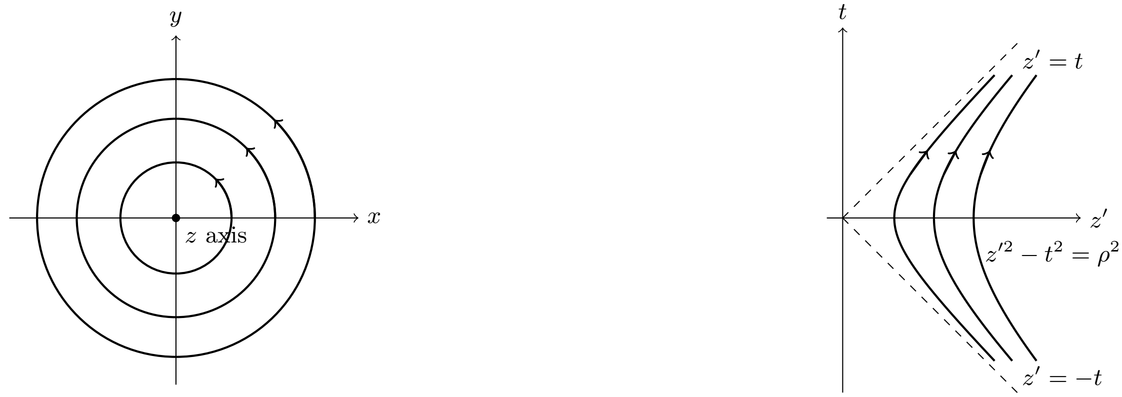
$$t = \frac{\mathrm{e}^{a\xi}}{a}\sinh(a\tau) \ , \qquad z' = \frac{\mathrm{e}^{a\xi}}{a}\cosh(a\tau) \ . \tag{3}$$

These coordinates automatically satisfy

$$z'^2 - t^2 = \frac{\mathrm{e}^{2a\xi}}{a^2} \ .$$

Thus a curve at fixed ξ is one of the hyperbolic trajectories discussed above. Comparing with $z'^2 - t^2 = \rho^2$, we have

$$\rho = \frac{\mathrm{e}^{a\xi}}{a} \ .$$



Rotating equilibrium
field lines of β^μ are circular

Accelerated equilibrium
field lines of β^μ are hyperbolae

FIG. 1: Schematic comparison between the two basic relativistic global-equilibrium configurations. For rigid rotation, the four-temperature field lines wind around the rotation axis. For accelerated equilibrium, the field lines are the orbits of a boost Killing vector and are hyperbolae in the (t, z') plane.

Therefore, fixed ξ means fixed proper acceleration,

$$\alpha(\xi) = \frac{1}{\rho} = a\mathrm{e}^{-a\xi} \ .$$

In particular, the observer at $\xi = 0$ has

$$\rho = \frac{1}{a} \ , \qquad \alpha = a \ .$$

Thus the parameter a is the proper acceleration of the reference Rindler observer located at $\xi = 0$. It is not the proper acceleration of the whole fluid.

The Minkowski metric becomes

$$\mathrm{d}s^2 = \mathrm{e}^{2a\xi}\left(\mathrm{d}\tau^2 - \mathrm{d}\xi^2\right) - \mathrm{d}x^2 - \mathrm{d}y^2 \ . \tag{4}$$

The coordinate τ is the natural time coordinate for accelerated observers, while ξ labels different accelerated worldlines.

Using (3), one finds

$$\frac{\partial t}{\partial \tau} = \mathrm{e}^{a\xi}\cosh(a\tau) = az' \ , \qquad \frac{\partial z'}{\partial \tau} = \mathrm{e}^{a\xi}\sinh(a\tau) = at \ .$$

Therefore,

$$\partial_\tau = a\left(z'\partial_t + t\partial_z\right) \ .$$

The boost Killing vector is thus

$$z'\partial_t + t\partial_z = \frac{1}{a}\partial_\tau \ .$$

Since the four-temperature field is

$$\beta = \frac{a}{T_0}\left(z'\partial_t + t\partial_z\right) \ ,$$

we obtain

$$\beta = \frac{1}{T_0}\partial_\tau \ .$$

This is the cleanest way to interpret accelerated global equilibrium. The state is stationary with respect to translations in the Rindler time τ, not with respect to translations in the inertial Minkowski time t.

The temperature profile in Rindler coordinates follows immediately. Since

$$z'^2 - t^2 = \frac{\mathrm{e}^{2a\xi}}{a^2} \ ,$$

we get

$$T(\xi) = \frac{T_0}{a\sqrt{z'^2 - t^2}} = T_0 \mathrm{e}^{-a\xi} \ .$$

Equivalently, using $\rho = \mathrm{e}^{a\xi}/a$, one can write

$$T(\rho) = \frac{T_0}{a\rho} \ .$$

This expression shows clearly that the proper temperature depends on which accelerated observer we consider, namely on the value of ξ or ρ. However, it does not depend on the Rindler time τ. Therefore each comoving accelerated observer measures a constant temperature.

An inertial observer at fixed Minkowski position z has a different description. In Minkowski coordinates the proper temperature field is

$$T(t, z) = \frac{T_0}{a\sqrt{(z + 1/a)^2 - t^2}} \ .$$

Thus an inertial observer at fixed z sees the local proper temperature of the fluid depend on time. This does not contradict equilibrium, because the inertial observer is not comoving with the fluid and is not following the boost Killing flow. At different Minkowski times, the observer samples different fluid elements belonging to different hyperbolae. The equilibrium is stationary only with respect to the Rindler time τ, namely along the accelerated flow.

To summarize, the accelerated equilibrium configuration is generated by a boost Killing vector. Its field lines are hyperbolae in Minkowski spacetime, corresponding to uniformly accelerated comoving observers. The parameter a is the proper acceleration of the reference observer at $\xi = 0$, for which the proper temperature is T_0. Other fluid elements have different proper accelerations and different constant proper temperatures. Therefore, global accelerated equilibrium is not characterized by a uniform proper acceleration, but by stationarity with respect to the boost Killing flow.

The general equilibrium state is given by a general combination of the rotation and acceleration configuration.

II. EXERCISE 4

From the definition of the local-equilibrium expectation value in linear response theory,

$$\langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{LE}} = \langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} + \Delta_{\mathrm{LE}} T^{\mu\nu}(x) \ . \tag{5}$$

We want to prove that the leading-order term coincides with the ideal-fluid form, namely

$$\langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} = \mathcal{A}(\beta^2)\beta^\mu(x)\beta^\nu(x) + \mathcal{B}(\beta^2) g^{\mu\nu} \ , \tag{6}$$

which is equivalent to

$$\langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} = (\varepsilon + p) u^\mu(x) u^\nu(x) - p g^{\mu\nu} \ , \tag{7}$$

where ε and p are the energy density and pressure, respectively.

The fundamental property we use to fix the form of an expectation value is the combination of the transformation properties of the operator we are computing and the symmetries of the state, namely of the density operator. First, the homogeneous equilibrium state is invariant under translations, which are represented on the Hilbert space by the operator

$$\widehat{\mathrm{T}}_y = \mathrm{e}^{\mathrm{i}\widehat{P}\cdot y} \ . \tag{8}$$

Under translations, the stress-energy tensor transforms as a local field:

$$\widehat{\mathrm{T}}_y \, \widehat{T}^{\mu\nu}(x) \, \widehat{\mathrm{T}}_y^{-1} = \widehat{T}^{\mu\nu}(x+y) \ . \tag{9}$$

Because of the homogeneity of the equilibrium state, we have

$$\widehat{\mathrm{T}}_y \, \widehat{\rho}_{\mathrm{GE}} \, \widehat{\mathrm{T}}_y^{-1} = \widehat{\rho}_{\mathrm{GE}} \ , \tag{10}$$

or, equivalently, the density operator commutes with the generators of translations. We have

$$\begin{aligned}
\langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} &= \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}} \, \widehat{T}^{\mu\nu}(x) \right] \\
&= \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}} \, \widehat{\mathrm{T}}_{-x}^{-1} \widehat{\mathrm{T}}_{-x} \, \widehat{T}^{\mu\nu}(x) \, \widehat{\mathrm{T}}_{-x}^{-1} \widehat{\mathrm{T}}_{-x} \right] \\
&= \mathrm{Tr}\left[\left(\widehat{\mathrm{T}}_{-x} \widehat{\rho}_{\mathrm{GE}} \, \widehat{\mathrm{T}}_{-x}^{-1} \right) \left(\widehat{\mathrm{T}}_{-x} \, \widehat{T}^{\mu\nu}(x) \, \widehat{\mathrm{T}}_{-x}^{-1} \right) \right] \ ,
\end{aligned}$$

where in the last equality we used the cyclic property of the trace. Then, using the transformation properties above, we get

$$\langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE}} = \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}} \, \widehat{T}^{\mu\nu}(0) \right] = \langle \widehat{T}^{\mu\nu}(0) \rangle_{\mathrm{GE}} \ .$$

Therefore, in homogeneous global equilibrium, the expectation value has no explicit dependence on the spacetime point x. In local equilibrium, instead, one promotes the constant thermodynamic parameters to slowly varying fields, such as $\beta^\mu = \beta^\mu(x)$ and $\zeta = \zeta(x)$, and the leading-order contribution is obtained by evaluating the global-equilibrium result locally.

Another property we must consider is how the expectation value transforms under a general Lorentz transformation. A Lorentz transformation Λ is represented on the Hilbert space by a corresponding unitary operator $\widehat{\Lambda}$. The stress-energy tensor operator transforms as

$$\widehat{\Lambda} \, \widehat{T}^{\mu\nu}(x) \, \widehat{\Lambda}^{-1} = (\Lambda^{-1})^{\mu}{}_{\lambda} (\Lambda^{-1})^{\nu}{}_{\sigma} \, \widehat{T}^{\lambda\sigma}(\Lambda x) \ , \tag{11}$$

where $\Lambda x^\alpha \equiv \Lambda^\alpha{}_\beta x^\beta$ is the Lorentz-transformed coordinate vector.

To understand how the global-equilibrium density operator transforms under a Lorentz transformation, we use the fact that a generic function of an operator is defined by its Taylor series. Hence,

$$\begin{aligned}
\widehat{\Lambda} \, \widehat{\rho}_{\mathrm{GE}} \, \widehat{\Lambda}^{-1} &= \frac{1}{Z} \widehat{\Lambda} \, \mathrm{e}^{-\beta \cdot \widehat{P}} \, \widehat{\Lambda}^{-1} \\
&= \frac{1}{Z} \widehat{\Lambda} \left(\widehat{\mathrm{I}} - \beta_\lambda \widehat{P}^\lambda + \frac{1}{2} \beta_{\lambda_1} \beta_{\lambda_2} \widehat{P}^{\lambda_1} \widehat{P}^{\lambda_2} - \dots \right) \widehat{\Lambda}^{-1} \\
&= \frac{1}{Z} \left(\widehat{\mathrm{I}} - \beta_\lambda \widehat{\Lambda} \, \widehat{P}^\lambda \, \widehat{\Lambda}^{-1} + \frac{1}{2} \beta_{\lambda_1} \beta_{\lambda_2} \widehat{\Lambda} \, \widehat{P}^{\lambda_1} \, \widehat{\Lambda}^{-1} \widehat{\Lambda} \, \widehat{P}^{\lambda_2} \, \widehat{\Lambda}^{-1} - \dots \right) \\
&= \frac{1}{Z} \mathrm{e}^{-\beta_\lambda \widehat{\Lambda} \widehat{P}^\lambda \widehat{\Lambda}^{-1}} \ .
\end{aligned}$$

Taking into account that $\widehat{P}^\lambda$ is a four-vector operator,

$$\widehat{\Lambda} \, \widehat{P}^\lambda \, \widehat{\Lambda}^{-1} = (\Lambda^{-1})^\lambda{}_\gamma \widehat{P}^\gamma \ ,$$

we get

$$\beta_\lambda \widehat{\Lambda} \, \widehat{P}^\lambda \, \widehat{\Lambda}^{-1} = \beta_\lambda (\Lambda^{-1})^\lambda{}_\gamma \widehat{P}^\gamma = (\Lambda\beta)_\gamma \widehat{P}^\gamma \ .$$

Therefore,

$$\widehat{\Lambda} \, \widehat{\rho}_{\mathrm{GE}}(\beta) \, \widehat{\Lambda}^{-1} = \widehat{\rho}_{\mathrm{GE}}(\Lambda\beta) \ .$$

Then

$$\begin{aligned}
\langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{GE},\beta} &= \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}(\beta) \, \widehat{T}^{\mu\nu}(x) \right] \\
&= \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}(\beta) \, \widehat{\Lambda}^{-1} \widehat{\Lambda} \, \widehat{T}^{\mu\nu}(x) \, \widehat{\Lambda}^{-1} \widehat{\Lambda} \right] \\
&= \mathrm{Tr}\left[\left(\widehat{\Lambda} \widehat{\rho}_{\mathrm{GE}}(\beta) \widehat{\Lambda}^{-1} \right) \left(\widehat{\Lambda} \, \widehat{T}^{\mu\nu}(x) \, \widehat{\Lambda}^{-1} \right) \right] \\
&= (\Lambda^{-1})^\mu{}_\sigma (\Lambda^{-1})^\nu{}_\gamma \, \mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}(\Lambda\beta) \, \widehat{T}^{\sigma\gamma}(\Lambda x) \right] \ .
\end{aligned}$$

Combining the last line with translation invariance, we can write

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\mathrm{GE},\beta} = (\Lambda^{-1})^{\mu}{}_{\sigma}(\Lambda^{-1})^{\nu}{}_{\gamma}\,\langle \widehat{T}^{\sigma\gamma}(x)\rangle_{\mathrm{GE},\,\Lambda\beta}\ . \tag{12}$$

Equivalently, this equation states that the expectation value is a rank-two tensor-valued function of the four-vector β^{μ}.

The above result means that we can compute the components of the expectation value in a preferred reference frame and then obtain the general expression by transforming back. The natural choice is the comoving reference frame, namely the frame where the fluid is at rest:

$$\Lambda\beta^{\sigma} = \frac{u_0^{\sigma}}{T} = \frac{1}{T}(1,\mathbf{0})^{\sigma}\ . \tag{13}$$

In this frame, the equilibrium density operator is invariant under ordinary spatial rotations. Therefore, the expectation value must be rotationally invariant. This implies that, in the comoving frame,

$$\langle \widehat{T}^{00}\rangle_{\mathrm{GE}} \neq 0\ , \qquad \langle \widehat{T}^{0i}\rangle_{\mathrm{GE}} = 0\ , \qquad \langle \widehat{T}^{ij}\rangle_{\mathrm{GE}} \propto \delta^{ij}\ . \tag{14}$$

Equivalently, in covariant language, the only available vector on which the expectation value can depend is the four-temperature β^{μ}. Since $\widehat{T}^{\mu\nu}$ is symmetric, the only possible symmetric rank-two tensors that can be constructed from β^{μ} and the metric are $\beta^{\mu}\beta^{\nu}$ and $g^{\mu\nu}$. Hence,

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\mathrm{GE}} = \mathcal{A}(\beta^2)\beta^{\mu}\beta^{\nu} + \mathcal{B}(\beta^2)g^{\mu\nu}\ . \tag{15}$$

Here the scalar coefficients can depend only on the Lorentz scalar $\beta^2 = \beta_{\mu}\beta^{\mu}$. Indeed they cannot depend on the coordinates given the homogeneity and being Lorentz scalars they can only depend on the proper Lorentz scalars I can build with the vectors and tensor at my disposal which for the case at hand is only $\beta_{\mu}\beta_{\nu}g^{\mu\nu}$. Also, since

$$\beta^{\mu} = \frac{u^{\mu}}{T}\ , \qquad \beta^2 = \frac{1}{T^2}\ ,$$

the expression (15) can be equivalently written as:

$$\langle \widehat{T}^{\mu\nu}(x)\rangle_{\mathrm{GE}} = \widetilde{\mathcal{A}}(\beta^2)u^{\mu}u^{\nu} + \mathcal{B}(\beta^2)g^{\mu\nu}\ , \tag{16}$$

where the new coefficients is a simply temperature rescaling of the original one

$$\widetilde{\mathcal{A}}(\beta^2) = \frac{\mathcal{A}(\beta^2)}{T^2} = \mathcal{A}(\beta^2)\beta^2\ .$$

Now we it is easy to prove that the two coefficients $\mathcal{A}$ and $\mathcal{B}$ are directly related with energy density and pressure and thus the expansion (15) is equivalent to the ideal form given in the hydrodynamic lectures. Starting from (15) and using $\beta^{\mu} = u^{\mu}/T$, we obtain:

$$\langle \widehat{T}^{\mu\nu}\rangle_{\mathrm{GE}} = \mathcal{A}(\beta^2)\beta^2 u^{\mu}u^{\nu} + \mathcal{B}(\beta^2)g^{\mu\nu}\ .$$

We now identify the energy density by projecting twice along the four-velocity:

$$\varepsilon = u_{\mu}u_{\nu}\langle \widehat{T}^{\mu\nu}\rangle_{\mathrm{GE}}\ .$$

Therefore,

$$\begin{aligned}\varepsilon &= u_{\mu}u_{\nu}\left[\mathcal{A}(\beta^2)\beta^{\mu}\beta^{\nu} + \mathcal{B}(\beta^2)g^{\mu\nu}\right]\\ &= \mathcal{A}(\beta^2)(u\cdot\beta)^2 + \mathcal{B}(\beta^2)u_{\mu}u^{\mu}\\ &= \mathcal{A}(\beta^2)\beta^2 + \mathcal{B}(\beta^2)\ .\end{aligned}$$

The pressure is obtained from the spatial projection:

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}\ . \tag{17}$$

Then:

$$p = -\frac{1}{3}\Delta_{\mu\nu}\langle \widehat{T}^{\mu\nu}\rangle_{\mathrm{GE}}\ .$$

Using $\Delta_{\mu\nu}u^\nu = 0$ and $\beta^\mu \propto u^\mu$, we have

$$\Delta_{\mu\nu}\beta^\mu\beta^\nu = 0 \ .$$

Moreover,

$$\Delta_{\mu\nu}g^{\mu\nu} = 3 \ .$$

Hence

$$p = -\frac{1}{3}\mathcal{B}(\beta^2)\Delta_{\mu\nu}g^{\mu\nu} = -\mathcal{B}(\beta^2) \ , \tag{18}$$

which finally leads to the identification:

$$\mathcal{B}(\beta^2) = -p \ , \tag{19}$$

and

$$\varepsilon = \mathcal{A}(\beta^2)\beta^2 - p \ . \tag{20}$$

The above can be expressed in the following equivalent way

$$\mathcal{A}(\beta^2)\beta^2 = \varepsilon + p \ .$$

so that substituting these identifications back into the covariant decomposition, we finally obtain

$$\langle \widehat{T}^{\mu\nu}\rangle_{\rm GE} = (\varepsilon + p)u^\mu u^\nu - p g^{\mu\nu} \ . \tag{21}$$

This is precisely the ideal-fluid form of the stress-energy tensor.

III. EXERCISE 5

In this exercise we analyze the properties of the dissipative corrections for the stress-energy tensor.
Considering only the four-temperature contribution we have:

$$\langle \widehat{T}^{\mu\nu}(x)\rangle \simeq \langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} + \int_\Omega {\rm d}^4y \int_0^1 {\rm d}z \ \left\langle \widehat{T}^{\mu\nu}(x),\, {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{T}^{\rho\sigma}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}} \partial_\rho\beta_\sigma(y) \ . \tag{22}$$

The correction depends on the myscroscopic correlator:

$$K^{\mu\nu,\rho\sigma}(y,x) \equiv \int_0^1 {\rm d}z \ \left\langle \widehat{T}^{\mu\nu}(x),\, {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{T}^{\rho\sigma}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}} \ , \tag{23}$$

so that the (22) can be equivalently written as:

$$\langle \widehat{T}^{\mu\nu}(x)\rangle \simeq \langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE} + \int_\Omega {\rm d}^4y\, K^{\mu\nu,\rho\sigma}(y,x)\,\partial_\rho\beta_\sigma(y) \ . \tag{24}$$

Now we want to prove that the correlator (23) defining the leading order correction to the stress-energy tensor actually depends on $y-x$. This is a direct consequence of the fact that the stress tensor is a local operator and that in linear response theory the expectation value is computed on the homogeneous equilibrium state.

To show this lets consider the definition of expectation value:

$$\begin{aligned}\left\langle \widehat{T}^{\mu\nu}(x),\, {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{T}^{\rho\sigma}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}} &= {\rm Tr}\left[\widehat{\rho}_{\rm GE}\,\widehat{T}^{\mu\nu}(x){\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{T}^{\rho\sigma}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right] \\ &\quad - \langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE}\langle {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\,\widehat{T}^{\rho\sigma}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\rangle_{\rm GE} \ .\end{aligned}$$

The second term, due to the ciclic property of the trace simply reduces to the disconnected part of $\widehat{T}(x)\widehat{T}(y)$ and thus at homogeneous global equilibrium is independent on x has we proved in the previous exercise. We can thus concentrate on the first part. Using the transformation law of the stress tensor under translation:

$$\widehat{\rm T}_y\,\widehat{T}^{\mu\nu}(x)\,\widehat{\rm T}_y^{-1} = \widehat{T}^{\mu\nu}(x+y) \ , \qquad \widehat{\rm T}_y = {\rm e}^{{\rm i}y\cdot\widehat{P}} \ , \tag{25}$$

we have:

$$\begin{aligned}\mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}\,\widehat{T}^{\mu\nu}(x)\mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\,\widehat{T}^{\rho\sigma}(y)\,\mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}}\right] &= \frac{1}{Z}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(x)\mathrm{e}^{z\beta\cdot\widehat{P}}\,\widehat{T}^{\rho\sigma}(y)\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right]\\ &= \frac{1}{Z}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\mathrm{e}^{\mathrm{i}x\cdot\widehat{P}}\widehat{T}^{\mu\nu}(0)\mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}}\mathrm{e}^{z\beta\cdot\widehat{P}}\,\widehat{T}^{\rho\sigma}(y)\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right]\\ &= \frac{1}{Z}\left[\mathrm{e}^{\mathrm{i}x\cdot\widehat{P}}\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(0)\mathrm{e}^{z\beta\cdot\widehat{P}}\,\mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}}\widehat{T}^{\rho\sigma}(y)\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right]\,,\end{aligned}$$

where we used that the various exponents of $\widehat{P}$ commute with each other. Now using the ciclic property of the trace we can move the factor $\exp(\mathrm{i}x\cdot\widehat{P})$ on the right and commuting again we get:

$$\begin{aligned}\frac{1}{Z}\left[\mathrm{e}^{\mathrm{i}x\cdot\widehat{P}}\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(0)\mathrm{e}^{z\beta\cdot\widehat{P}}\,\mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}}\widehat{T}^{\rho\sigma}(y)\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right] &= \frac{1}{Z}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(0)\mathrm{e}^{z\beta\cdot\widehat{P}}\,\mathrm{e}^{-\mathrm{i}x\cdot\widehat{P}}\widehat{T}^{\rho\sigma}(y)\mathrm{e}^{\mathrm{i}x\cdot\widehat{P}}\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right]\\ &= \frac{1}{Z}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(0)\mathrm{e}^{z\beta\cdot\widehat{P}}\,\widehat{T}^{\rho\sigma}(y-x)\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right]\,,\end{aligned}$$

that is:

$$\mathrm{Tr}\left[\widehat{\rho}_{\mathrm{GE}}\,\widehat{T}^{\mu\nu}(x)\mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\,\widehat{T}^{\rho\sigma}(y)\,\mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}}\right] = \frac{1}{Z}\left[\mathrm{e}^{-\beta\cdot\widehat{P}}\,\widehat{T}^{\mu\nu}(0)\mathrm{e}^{z\beta\cdot\widehat{P}}\,\widehat{T}^{\rho\sigma}(y-x)\,\mathrm{e}^{-z\beta\cdot\widehat{P}}\right]\,,$$

which together with the constancy of the disconnected part implies:

$$K^{\mu\nu,\rho\sigma}(y,x) = K^{\mu\nu,\rho\sigma}(y-x)\,. \tag{26}$$

Note that replacing the first $\widehat{T}^{\mu\nu}(x)$ with a generic local operator the corresponding correlator would still only depend on $y-x$.