Hydrodynamics in heavy-ion collisions

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Chapter 1

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1.1 Heavy-Ion Collisions (HIC)

Experiments at the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC) collide heavy nuclei, such as Au^{197} or Pb^{208} at center-of-mass (c.o.m.) energies up to

$$\sqrt{s_{NN}} \simeq 2\sqrt{E_A^N E_B^N} \sim 5\text{TeV} \sim 5000 \text{ m}_\text{N}$$

per nucleon pair, where $E_{A/B}^N$ denote the energies of nucleons in the center-of-mass frame, and $m_N \simeq 1\text{GeV}$ is the rest mass of a nucleon. Central objective of such experiments is to create a hot state of strong-interaction matter, which is called the Quark-Gluon Plasma (QGP). Based on first principle calculations in the underlying theory of Quantum Chromodynamcis (QCD), such a state exists for temperatures $T \gtrsim T_c$ exceeding the cross-over temperature $T_c \simeq 155\text{MeV} \simeq 1.8 \times 10^{12} K$, which in heavy-ion collision experiments can be realized for a short period of time $\sim 10\text{fm}/c$, where

$$1\text{fm}/c \simeq 3.310^{-24} s$$

Due to the inherently short time scales involved, experimental measurements in heavy-ion collisions are only possible in the final state after the highly complicated reaction dynamics takes place. Hence to infer equilibrium and non-equilibrium properties of QCD matter from heavy-ion collisions, a robust theoretical understanding of the underlying space-time dynamics is necessary.

Since a first principle description of the dynamics of a heavy-ion collisions from the underlying theory of QCD remains an outstanding problem in theoretical physics, one instead resorts to a sequence of different effective descriptions of QCD which exploit a separation of time scales in the reaction dynamics.

- Colliding Nuclei & Initial state energy deposition:
- Pre-equilibrium dynamics
- Hydrodynamic expansion
- Hadronization
- Hadronic re-scattering & freeze-out

State-of-the art simulations of heavy-ion collisions typically consist of a sequence of software tools to describe the dynamics during all these different stages of a heavy-ion collision. Based on this standard model of heavy-ion collisions, viscous relativistic hydrodynamics provides the central tool to describe the space-time dynamics of the QGP from a macroscopic perspective. Non-equilibrium QCD processes are important to link the cold QCD properties of the colliding nuclei to the initial state for the hydrodynamic expansion of the QGP (hot QCD)

1.2 Hydrodynamics in heavy-ion collisions

Hydrodynamics provides a universal long-energy effective theory for the long time, long-wavelength description of an interacting QFT. Since the QGP created in heavy-ion collisions is extremely small, typically on the order of the size of an atomic nucleus $\sim$ 10fm or smaller, and very short lived, typically $\lesssim$ 10fm/c heavy-ion collisions provide extreme conditions pushing the theory to the limits of its range of applicability and possibly beyond. Hence an important line of research concerns the question when and under what conditions hydrodynamics is applicable to describe the space-time dynamics of the QGP in heavy-ion collisions.

We recapitulate, that hydrodynamics provides an effective macroscopic description in terms of the conserved quantities of the system, which in the case of QGP correspond to the

- energy-momentum tensor $T^{\mu\nu}$
- conserved currents J_i^μ for light-flavor quarks $i = u, d, s$[1]

Since these represent conserved quantities, their evolution is governed by conservation laws

$$\boldsymbol{\nabla}_\mu T^{\mu\nu} = 0 \,, \qquad \boldsymbol{\nabla}_\mu J^\mu = 0 \,. \tag{1.1}$$

However, as the above equations represent an underdetermined system of equations, they need to be closed by constitutive equations which provide explicit expressions for $T^{\mu\nu}$ and J^μ as a function of hydrodynamic variables in an expansion around local thermal equilibrium. While the conservation equations remain valid under all circumstances in QCD, it is the constitutive equations that are only satisfied if the system is sufficiently close to thermal equilibrium. Understanding the applicability of hydrodynamics in heavy-ion collision thus requires understanding when and to what degree constitutive equations are satisfied.

Before we further address this question, let us briefly investigate how this works out if we consider an ideal hydrodynamic description, which describes the evolution of a system that is considered in equilibrium at every point in space-time. In this case the state of the system is characterized by

- local rest-frame velocity $u^\mu(x)$
- local temperature $T(x)$
- local chemical $\mu(x)$

and the energy-momentum tensor and conserved currents take the form

$$T^{\mu\nu} = e u^\mu u^\nu + p\Delta^{\mu\nu} \qquad J_i^\mu = n_i u^\mu \,, \tag{1.2}$$

where energy density e, pressure p and charge densities n_i are determined from equilibrium QCD thermodynamics, i.e. from the equations of state

$$e(x) = e\left(T(x), \{\mu_j(x)\}\right) \,, \quad p(x) = p\left(T(x), \{\mu_j(x)\}\right) \,, \quad n_i(x) = n_i\left(T(x), \{\mu_j(x)\}\right) \,, \tag{1.3}$$

Since in high-energy heavy-ion collisions the net charge densities are typically very small, leading to $\mu_j/T \ll 1$, these can be determined from first principles lattice QCD calculations. In a high-temperature QGP, one then finds that to rather good approximation

$$e(T, \mu_j \ll T) \simeq e(T) = \frac{\pi^2}{30}\nu_{\text{eff}}(T)T^4 \,, \qquad p(T, \mu_j \ll T) \simeq \frac{e(T)}{3} \tag{1.4}$$

which are characteristic for a conformal system, with an effective number of bosonic degrees of freedom $\nu_{\text{eff}}(T \gg T_c) \approx 40$.

We note that in practice it is often times sufficient to consider $\mu_j = 0$ and only focus on the evolution of the energy-momentum tensor and we will adopt this idea throughout the following to avoid some unnecessary complications.

Slides

[1] Note that alternatively, one can also perform a change of basis and instead consider the linear combinations of baryon number B, electric charge Q and strangeness S as frequently done in the literature.

1.3 Hydrodynamics for (generalized) Bjorken flow

Now to understand the applicability of hydrodynamics in heavy-ion collisions, we first focus on the early stages of the collision, where immediately after the collision the system is subject to rapid longitudinal expansion, where the QGP created in the collision expands into the space left behind by the expanding nuclei. Since matter is created locally at rest in the transverse plane, we may neglect the transverse expansion for a short period of time, as it needs time to be generated dynamically.

Central objective is then to predict how the energy deposited in the actual collision is converted into thermal energy of the hydrdynamic QGP at the onset o the hydrodynamic expansion.

1.3.1 Generalized Bjorken flow

If we neglect the transverse dynamics during the pre-equilibrium phase, it is sufficient to consider a transversely homogenous system, such that in Minkowski coordinates the energy momentum tensor

$$T^{\mu\nu}(t, x_T, z) = T^{\mu\nu}(t, z) \tag{1.5}$$

is translation invariant in the transverse (x_T) plane.

Evidently along the longitudinal direction, the system is not translation invariant as the collision takes place at a well defined collision point. However, in high-energy kinematics the system, approximately becomes independent under boosts along the longitudinal direction, as we can see from the following consideration. If we consider the high-energy limit, nuclei are highly Lorenz contracted and all matter is produced from points $(t, z) = (0, 0)$ in space-time. Now if we observe the QGP at points (t, z) in the foward light-cone, the matter originating from the collision point will move towards us with a collective velocity

$$v_z = \frac{z}{t}\,, \qquad \gamma = \frac{1}{\sqrt{1 - z^2/t^2}} \tag{1.6}$$

such that the natural rest-frame of the system is given by

$$u^\mu = \begin{pmatrix} \gamma \\ \gamma v_z \end{pmatrix} = \begin{pmatrix} \frac{t}{\sqrt{t^2 - z^2}} \\ \frac{z}{\sqrt{t^2 - z^2}} \end{pmatrix} \tag{1.7}$$

in Minkowski coordinates. Since the description in Minkowksi space appears rather complicated already at this level, it is advantageous to use a different set of coordinates, which are better suited to describe the invariance under longitudinal boosts. These are the Milne coordinates

$$\text{proper time: } \tau = \sqrt{t^2 - z^2}\,, \qquad \text{(invariant under long. boosts)} \tag{1.8}$$

$$\text{space-time rapidity: } \eta = \operatorname{atanh}(z/t) \qquad \text{(additive under long. boosts)} \tag{1.9}$$

and the inverse transformation is given by

$$t = \tau \cosh(\eta)\,, \qquad z = \tau \sinh(\eta)\,. \tag{1.10}$$

such that the rest-frame velocity of the system in Minkowksi coordinates can be expressed as

$$u^\mu = \begin{pmatrix} \cosh(\eta) \\ \sinh(\eta) \end{pmatrix} = \begin{pmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.11}$$

which as we see from the second expression, corresponds to system at rest, that is Lorentz boosted by a rapidity equal to the space-time rapidity η.

Now if we use the Milne coordinates, as our new coordinates in the sense of general relativity, we also need to transform the rest-frame velocity as a four vector, i.e. we obtain

$$\bar{u}^\mu = \left(\bar{u}^\tau, \bar{u}^x, \bar{u}^y, \bar{u}^\eta\right)^T \tag{1.12}$$

where the vector transforms as

$$\bar{u}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} u^\nu \tag{1.13}$$

we have that by means of the Jacobian

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial z} \\ \frac{\partial \eta}{\partial t} & \frac{\partial \eta}{\partial z} \end{pmatrix} = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ \frac{-\sinh(\eta)}{\tau} & \frac{\cosh(\eta)}{\tau} \end{pmatrix} \tag{1.14}$$

the rest frame in Milne coordinates simply becomes

$$\bar{u}^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{1.15}$$

i.e. in these co-moving coordinates we are describing a system that is locally at rest. Evidently, we also need to transform the metric, according to the general transformation rule

$$\bar{g}_{\rho\sigma} = \frac{\partial x^\mu}{\partial \bar{x}^\rho} \, g_{\mu\nu} \, \frac{\partial x^\nu}{\partial \bar{x}^\sigma} \tag{1.16}$$

yielding the following result for the metric

$$\bar{g}_{\mu\nu} = \mathrm{diag}(1, -1, -1, -a(\tau)^2) , \tag{1.17}$$

and its inverse

$$\bar{g}^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1/a(\tau)^2) , \tag{1.18}$$

where we introduced the scale factor $a(\tau)$, which is

$$a(\tau) = \tau \tag{1.19}$$

for Bjorken flow. We see at this stage that the pre-equilibrium QPG is locally at rest in a space-time that is anistropically expanding at a rate $\frac{\dot{a}}{a} = \frac{1}{\tau}$, i.e. very rapidly at early times and more an more slowly at later times.

Due to the expansion the energy density of the system will be diluted and the system will cool. Moreover, as the expansion is anisotropic, as the system initially only expands along the longitudinal direction, this will inevitably induce anisotropies in the system which can only be erased due to interactions. Hence it is the interplay of the anisotropic expansions and interactions that determines the evolution of the early pre-equilibirum QGP, and the applicability of hydrodynamics to describe it.

1.3.2 Evolution of the energy-momentum tensor

The evolution of the energy momentum tensor in (generalized) is governed by the conservation equations

$$\boldsymbol{\nabla}_\mu T^{\mu\nu} = 0 , \tag{1.20}$$

where the covariant derivative of the energy-momentum tensor is given by

$$\boldsymbol{\nabla}_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} T^{\mu\nu} \right) + \Gamma^\nu_{\mu\rho} T^{\mu\rho} \tag{1.21}$$

where we introduced the metric determinant and Christoffel

$$\text{Metric determinant :} \quad \sqrt{-g} = \sqrt{-\det(g_{\mu\nu})} = a(\tau) \tag{1.22}$$

$$\text{Christoffel Symbols:} \quad \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\frac{\partial g_{\mu\beta}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu} \right) \tag{1.23}$$

and the non-vanishing Christoffel symbols in generalized Bjorken flow are

$$\Gamma^\eta_{\eta\tau} = \frac{1}{2} g^{\eta\eta} \frac{\partial g^{\eta\eta}}{\partial \tau} = \frac{\dot{a}}{a} \tag{1.24}$$

$$\Gamma^\eta_{\tau\eta} = \frac{1}{2} g^{\eta\eta} \frac{\partial g^{\eta\eta}}{\partial \tau} = \frac{\dot{a}}{a} \tag{1.25}$$

$$\Gamma^\tau_{\eta\eta} = -\frac{1}{2} g^{\tau\tau} \frac{\partial g^{\eta\eta}}{\partial \tau} = \dot{a} a \tag{1.26}$$

Now if we consider the conservation equations, we can further take advantage of the symmetries of the (generalized) Bjorken flow, which are

- boost invariance in the long. direction
- translation invariance in the transverse plane
- isotropy in the transverse plane

such that the energy-momentum tensor takes the generic form

$$T^{\mu}_{\ \nu} = \text{diag}(T^{\tau}_{\ \tau}, T^{x}_{\ x}, T^{y}_{\ y}, T^{\eta}_{\ \eta}) \tag{1.27}$$

where physically, the different components correspond to the local energy density e, transverse pressure p_T and longitudinal pressure p_L, according to

$$T^{\tau}_{\ \tau} = e \ , \quad T^{x}_{\ x} = T^{y}_{\ y} = -p_T \ , \quad T^{\eta}_{\ \eta} = -p_L \tag{1.28}$$

such that for a conformal system

$$T^{\mu}_{\ \mu} = e - 2p_T - p_L = 0 \ . \tag{1.29}$$

Now if we consider the conservation equations, then due to the assumed symmetries derivatives $\partial_x, \partial_y, \partial_\eta$ vanish identically and the only interesting equation is the time like one ($\nu = \tau$), which takes the form

$$\frac{1}{a(\tau)}\partial_\tau\Big(a(\tau)T^{\tau\tau}\Big) = -\dot{a(\tau)}a(\tau)T^{\eta\eta} \tag{1.30}$$

such that upon re-arrangements we obtain

$$\partial_\tau T^{\tau\tau} = -\frac{\dot{a}(\tau)}{a(\tau)}\Big(T^{\tau\tau} - T^{\eta}_{\ \eta}\Big) \tag{1.31}$$

Specifically, if we consider the usual Bjorken flow and express the conservation law in terms of physical quantities, we obtain

$$\partial_\tau e = -\frac{e + p_L}{\tau} \tag{1.32}$$

where the first term describes the dilution of the energy density due to the increase of the volume, while the second term describes the work performed against the longitudinal expansion of the system. We note that we can obtain a very similar results if we consider standard thermodynamics, where

$$dU = -pdV \ , \quad U = eV \ , \quad dU = edV + Vde \ , \Rightarrow deV = -(e+p)dV \tag{1.33}$$

such that upon using

$$\frac{1}{V}\frac{dV}{d\tau} = \frac{\dot{a}}{a} \tag{1.34}$$

we obtain

$$\partial_\tau e = -\frac{\dot{a}}{a}(e+p) \ . \tag{1.35}$$

which is of the same form as (1.32), however as we assumed the system to be in equilibrium, the longitudinal pressure p_L is replaced with the equilibrium thermodynamic pressure p.

Evidently the conservation law alone is not sufficient to predict the evolution and we need additional information about the longitudinal pressure to describe the evolution of the energy density of the system, which can come from

- Microscopic calculations (e.g. in kinetic theory)
- Hydrodynamic constitutive equations

Hence understanding the applicability of hydrodydanmics always reduces to the question wether or not constitutive equations are (approximately) satisfied.

1.3.3 Hydrodynamic constitutive equations

Ideal hydrodynamics: We start with the constitutive equations in in ideal hydrodynamics and for simplicity consider a conformal fluid. Since in ideal hydrodynamics the system is assumed to be in local thermal equilibrium, we have

$$p_T = p_L = p \tag{1.36}$$

and for a conformal system

$$T^{\mu}_{\ \mu} = e - 2p_T - p_L = 0 \ , \Rightarrow p = e/3 \ . \tag{1.37}$$

The energy conservation equation then takes the form

$$\partial_\tau e = -\frac{4}{3}\frac{e}{\tau} \tag{1.38}$$

such that for ideal Bjorken flow, the energy density behaves as

$$e(\tau) = \frac{e_0 \tau_0^{4/3}}{\tau^{4/3}} \ . \tag{1.39}$$

Navier Stokes: If we consider the first order dissipative corrections for (generalized) Bjorken flow, we still have

$$T^{\mu}_{\ \nu} = (e, -p_T, -p_T, -p_L) \ , \tag{1.40}$$

but now p_T and p_L receive dissipative corrections due to longitudinal shear induced by the expansion. Specifically, we have

$$p_L = -T^{\eta}_{\eta} = p - \pi^{\eta}_{\ \eta} \qquad p_T = -T^{x}_{x} = -T^{y}_{\ y} = p - \pi^{x}_{\ x} \tag{1.41}$$

such that for a conformal system with $T^{\mu}_{\ \mu} = 0$, it follows

$$p_T = p + \frac{1}{2}\pi^{\eta}_{\ \eta} \tag{1.42}$$

such that the long. shear stress $\pi^{\eta}_{\ \eta}$ introduces a pressure anisotropy. Now in first order hydrodynamics, the shear stress tensor $\pi^{\mu\nu}$ satisfies the constitutive equation

$$\pi^{\mu\nu}_{NS} = 2\eta\sigma^{\mu\nu} \tag{1.43}$$

where explicitly

$$\sigma^{\mu\nu} = \frac{1}{2}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\nu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right)\boldsymbol{\nabla}_\alpha u_\beta \tag{1.44}$$

with $\boldsymbol{\nabla}_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^{\mu}_{\alpha\beta}u_\mu$. Since the fluid is at rest in the comoving coordinates the only non-trivial contributions is due to

$$\boldsymbol{\nabla}_\eta u_\eta = -\Gamma^{\tau}_{\eta\eta}u_\tau = -\dot{a}a \tag{1.45}$$

Now using $\Delta^{\eta\eta} = g^{\eta\eta} = -1/a^2$, we can then obtain long. shear stress as

$$\sigma^{\eta\eta} = -\frac{2}{3}\frac{\dot{a}}{a^3} \tag{1.46}$$

such that eventually

$$\pi^{\eta}_{\ \eta}\Big|_{NS} = \frac{4}{3}\eta\frac{\dot{a}}{a} \ . \tag{1.47}$$

Specifically, for the usual Bjorken flow the longitudinal pressure becomes

$$p_L^{NS} = p - \frac{4}{3}\frac{\eta}{\tau} \tag{1.48}$$

and we see that the longitudinal expansion leads to a reduction of the longitudinal pressure.

Since the viscosity is dimensionful quantity, it is useful to re-express it using thermodynamic relations, as

$$\eta = \frac{\eta T}{e+p}\frac{e+p}{T} \overset{Ts=e+p}{=} \frac{\eta}{s}\frac{e+p}{T} \overset{p=e/3}{=} \frac{4}{3}\frac{\eta}{s}\frac{e}{T} \tag{1.49}$$

such that the first order constitutive equation for the long. pressure in conformal Bjorken flow takes the form

$$\frac{p_L^{NS}}{e} = \frac{1}{3} - \frac{16}{9}\frac{\eta}{s}\frac{1}{T(\tau)\tau} \tag{1.50}$$

Since $T(\tau)\tau$ increases as a function of time, the solution eventually converges towards the ideal hydrodynamic constitutive relation for $T(\tau)\tau \gg 1$. However, we also see that at early times $T(\tau)\tau \ll 1$ the long. pressure can become negative, signaling an unphysical behavior that would indicate a mechanical instability of the system. Evidently, this indicates a breakdown of the validity of this first order dissipative hydrodynamics description.

Israel Stuart hydrodynamics: Since the first order hydrodynamic theory is not applicable at early times, to what extent this can be attributed to the structural deficiencies of the first order Navier-Stokes theory, and one may wonder what improved hydrodynamic descriptions such as Isreal-Stuart hydrodynamics predict. In Israel-Stuart hydrodynamics the shear-stress tensor $\pi^{\eta}_{\ \eta}$ relaxes to its Navier-Stokes value $\pi^{\eta}_{\ \eta}\Big|_{NS}$ on a time scale τ_π as described by the relaxation equation

$$\partial_\tau \pi^{\eta}_{\ \eta} = -\frac{1}{\tau_\pi}\Big(\pi^{\eta}_{\ \eta} - \pi^{\eta}_{\ \eta}\Big|_{NS}\Big). \tag{1.51}$$

Strikingly, the combined set of evolution equations for the energy density e and shear-stress tensor

$$\partial_\tau e = -\frac{e+p_L}{\tau}\,, \qquad p_L = e/3 - \pi^{\eta}_{\ \eta}\,, \tag{1.52}$$

$$\partial_\tau \pi^{\eta}_{\ \eta} = -\frac{1}{\tau_\pi}\Big(\pi^{\eta}_{\ \eta} - \frac{4}{3}\frac{\eta}{\tau}\Big). \tag{1.53}$$

shows a very interesting behavior, which features not only the emergence of the Navier-Stokes hydrodynamic limit at late times $\tau/\tau_\pi(\tau) \gg 1$ but also a stable fixed point at early times $\tau/\tau_\pi(\tau) \ll 1$. By looking at this theory, we therefore realize that relativistic hydrodynamic theories, such as Israel-Stuart theory, describe not only the dynamics close to equilibrium, but when pushed beyond near-equilibrium situations also model the approach towards equilibrium. However, the solutions still feature an unphysical negative longitudinal pressure at early times, such that in order to describe the dynamics of the system, we need to resort to a more microscopic description.

1.3.4 Early time dynamics in kinetic theory

We can obtain a more microscopic description of the early time dynamics in kinetic theory, where in general coordinates, the phase-space distribution $f(x,p)$ of (on-shell) quasi-particles at positions $x = x^\mu$ with momenta $p = p^i$, satisfies the Boltzmann equation

$$p^\mu \partial_\mu f(x,p) - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = C[f](x,p) \tag{1.54}$$

with the microphysics encoded in the collision kernel $C[f](x,p)$. Generally, the ides is then that by solving this partial integro-differntial, we can obtain the evolution of the phase-space distribution $f(x,p)$ and subsequently infer macroscopic properties of the system, such as the energy-momentum tensor

$$T^{\mu\nu}(x) = \nu_{\text{eff}} \int \frac{d^3p}{(2\pi)^3} \sqrt{-g(x)}\, \frac{p^\mu p^\nu}{E_p}\, f(x,p) \tag{1.55}$$

Specifically, in order to study the evolution of the pressure anisotropy in a (generalized) Bjorken flow, we need to calculate the longitudinal pressure

$$p_L(\tau) = -T^\eta_{\ \eta}(\tau) = \nu_{\text{eff}} \int \frac{d^2p_T}{(2\pi)^2} \int \frac{dp^\eta}{(2\pi)}\, a(\tau)\, \frac{a(\tau)^2 p^\eta}{E_p}\, f(x,p_T,p^\eta)\,. \tag{1.56}$$

However, it is more advantageous at this point, to re-parametrize the longitudinal momentum dependence in terms of the momentum variable

$$p_\parallel = a(\tau) p^\eta\,, \qquad p_\parallel^2 = -p_\eta p^\eta \tag{1.57}$$

which carries the dimension of momentum, such that in terms of this new variable, the longitudinal pressure takes the more intuitive form

$$p_L(\tau) = \nu_{\text{eff}} \int_p \frac{p_\parallel^2}{E_p}\, f(x,p_T,p_\parallel)\,, \qquad \int_p \equiv \int \frac{d^2p_T}{(2\pi)^2} \int \frac{dp_\parallel}{2\pi} \tag{1.58}$$

Now in order to express the Boltzmann equation in these coordinates, we obviously need to evaluate

$$\Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = \left(\Gamma^\eta_{\tau\eta} + \Gamma^\eta_{\eta\tau}\right) p^\eta p^\tau \frac{\partial f}{\partial \eta} = 2\frac{\dot a}{a} p^\tau p^\eta \frac{\partial f}{\partial p^\eta} = 2\frac{\dot a}{a} p^\tau p_\parallel \frac{\partial f}{\partial p_\parallel}. \tag{1.59}$$

However, when changing momentum variables from p^η to $p_\parallel = a(\tau)p^\eta$, we also need to consider the fact that time derivatives in the Boltzmann equation are taken at constant p^η, such that in order to re-formulate the evolution equation at constant $p_\parallel$, we also need to re-express[2]

$$p^\tau \partial_\tau f\Big|_{p^\eta} = p^\tau \partial_\tau f\Big|_{p_\parallel} + p^\tau \frac{\partial p_\parallel}{\partial \tau}\Big|_{p^\eta} \frac{\partial f}{\partial p_\parallel}\Big|_\tau = p^\tau \frac{\partial f}{\partial \tau}\Big|_{p_\parallel} + \frac{\dot a}{a} p^\tau p_\parallel \frac{\partial f}{\partial p_\parallel}\Big|_\tau \tag{1.60}$$

which partially cancels against the term in (1.59). By collecting everything, and dividing by $p^\tau = E_p$ the Boltzmann equation for generalized Bjorken flow takes the form

$$\partial_\tau f(\tau,p) = \underbrace{\frac{\dot a}{a} p_\parallel \partial_{p_\parallel} f(\tau,p)}_{\text{longitudinal expansion}} + \underbrace{\frac{1}{E_p} C[f](\tau,p)}_{interactions}\,. \tag{1.61}$$

Relaxation time approximation

Evidently, to infer any physical properties, we now need to specify the collision kernel. We consider the relaxation time approximation (RTA), where the collision kernel is approximated as

$$C[f](x,p) = -\frac{p^\mu u_\mu(x)}{\tau_R(x)} \left(f(x,p) - f_{\text{eq}}(x,p)\right)$$

[2] We can easily see this by equating the differentials $df = \frac{\partial f}{\partial \tau}\Big|_{p^\eta} d\tau + \frac{\partial f}{\partial \tau}\Big|_\tau dp^\eta$ and $df = \frac{\partial f}{\partial \tau}\Big|_{p_\parallel} d\tau + \frac{\partial f}{\partial \tau}\Big|_\tau dp_\parallel$ after setting $dp^\eta = 0$, such that the derivative $\frac{\partial p_\parallel}{\partial \tau}$ is taken at constant p^η.

which is a particularly simple form of the collision kernel, where all deviations from local thermal equilibrium are assumed to decay on a characteristic timescale of the inverse of the relaxation time $1/\tau_R$. Here the equilibrium distribution is given by

$$f_{\rm eq}(x,p) = f_{\rm Bose-Einstein}\left(\frac{p^\mu u_\mu(x)}{T(x)})\right)$$

with the local rest-frame velocity $u^\mu(x)$ and temperature $T(x)$ determined from the Landau matching condition

$$T^\mu_{\ \nu}(x)u^\nu(x) = e(x)u^\mu(x)\ , \qquad e(x) = \frac{\pi^2}{30}\nu_{\rm eff}T^4(x) \tag{1.62}$$

which is necessary to satisfy energy-momentum conservation. If we consider a conformal system, the relaxation time τ_R is determined by the temperature of the system, such that

$$\tau_R(x) = 5\frac{\eta/s}{T(x)}\ , \qquad \eta/s = const \tag{1.63}$$

and we also note for completeness, that the second order transport coefficient $\tau_\pi = \tau_R$, such that $T\tau_\pi = 5\eta/s$.

Bjorken flow

Now if we consider the conventional Bjorken flow, the Boltzmann equation (1.61) in relaxation time approximation takes the explicit form

$$\partial_\tau f(\tau,p) = \frac{1}{\tau}p_\parallel\partial_{p_\parallel}f(\tau,p) - \frac{1}{\tau_R(\tau)}\Big(f(\tau,p) - f_{\rm eq}(\tau,p)\Big) \tag{1.64}$$

which clearly shows the competition between the longitudinal expansion, with rate $\frac{\dot a}{a} = \frac{1}{\tau}$ and interactions with rate $\frac{1}{\tau_R(\tau)}$. Since at early times the longitudinal expansion is extremely rapid, it dominates the dynamics and drives the system away from thermal equilibrium. However, as the system evolves the longitudinal expansion rate becomes smaller and smaller, such that eventually interactions dominate and drive the system towards equilibrium. We can therefore anticipate two different regimes, namely

- Early times ($\tau \ll \tau_R$): Expansion dominated
- Late times ($\tau \ll \tau_R$): Interaction dominated

for which we can obtain approximate solutions to the Boltzmann equation.

Late times ($\boldsymbol{\tau \ll \tau_R}$)

Since interactions dominate, we can perform a standard expansion of the kinetic equation around local thermal equilibrium, which simply yields the hydrodynamic constitutive equation

$$p_L/e = \frac{1}{3} - \frac{16}{9}\frac{\eta/s}{T(\tau)\tau} \overset{\eta/s=\frac{T\tau_R}{5}}{=} \frac{1}{3} - \frac{16}{45}\frac{\tau_R(\tau)}{\tau} \tag{1.65}$$

and formally corresponds to an expansion in powers of τ_R/τ, i.e. the ratio of the longitudinal expansion rate, to the interaction rate.

Early times

Since at early times the dynamics of the system is dominated by the longitudinal expansion, we may obtain an approximate solution, by expanding the solution in the number ob interactions, or more formally in powers of τ/τ_R i.e. the ratio of the interaction rate to the expansion rate.
Free streaming solution: In the leading order of this expansion, we may neglect interactions altogether, resulting in the free-streaming Boltzmann equation

$$\partial_\tau f(\tau,p) = \frac{1}{\tau}p_\parallel\partial_{p_\parallel}f(\tau,p) \tag{1.66}$$

which is a linear first order PDE and can thus be solved immediately by the method of characteristics, and the solution is given by

$$f_{FS}(\tau,p_T,p_\parallel) = f_0(p_T,\frac{\tau}{\tau_0}p_\parallel)\ , \tag{1.67}$$

for an initial condition $f(\tau_0, p_T, p_\parallel) = f_0(, p_T, p_\parallel)$, as we may also verify by differentiation.

Since we ignored interactions, the transverse momentum (p_T) distribution remains unchanged, however as a consequence of the longitudinal expansion, the longitudinal momentum distribution gets squeezed by a factor τ_0/τ, such that for times $\tau > \tau_0$, the longitudinal pressure decreases. We can see this explicitly by calculating the long. pressure according to

$$p_L^{FS}(\tau) = -T^\eta_{\ \eta}(\tau) = \nu_{\text{eff}} \int_p \frac{p_\parallel^2}{\sqrt{p_T^2 + p_\parallel^2}}\ f_0(p_T, \frac{\tau}{\tau_0} p_\parallel) \tag{1.68}$$

such that upon changing variables to

$$q_\parallel = \frac{\tau}{\tau_0}\ , \qquad dp_\parallel = \frac{\tau_0}{\tau} dq_\parallel \tag{1.69}$$

we obtain

$$p_L^{FS}(\tau) = \nu_{\text{eff}} \int_q \left(\frac{\tau_0}{\tau}\right)^3 \frac{q_\parallel^2}{\sqrt{q_T^2 + \left(\frac{\tau_0}{\tau}\right)^2 q_\parallel^2}}\ f_0(q_T, q_\parallel) \tag{1.70}$$

whereas the energy density behaves as

$$e^{FS}(\tau) = \nu_{\text{eff}} \int_q \left(\frac{\tau_0}{\tau}\right) \sqrt{q_T^2 + \left(\frac{\tau_0}{\tau}\right)^2 q_\parallel^2}\ f_0(q_T, q_\parallel) \tag{1.71}$$

such that for $\tau \gg \tau_0$ we have

$$p_L^{FS} \sim \left(\frac{\tau_0}{\tau}\right)^3\ , \qquad e^{FS} \sim \left(\frac{\tau_0}{\tau}\right)\ , \tag{1.72}$$

and the longitudinal pressure becomes much smaller than the energy density, due to the effects of the rapid longitudinal expansion. Noteably, this happens regardless of the initial condition, indicating that even in the absence of interaction the longitudinal expansion leads to a partial memory loss about the initial conditions. Such behavior, where solutions for different initial conditions converge to the same result is usually referred to as an attractor behavior, and in the concrete case it is the early-time/free-streaming attractor.

Since τ_0 is an any case an artificial scale, we are interested in taking the limit $\tau_0 \to 0$, which amounts to describing the dynamics starting immediately after the collision. In this limit the solution for the phase-space distribution is given by

$$f^{FS}(\tau, p_T, p_\parallel) = f_0(p_T)\delta(\tau p_\parallel) \tag{1.73}$$

which has vanishing longitudinal pressure and an energy density that decays proportional to $\sim 1/\tau$

$$p_L^{FS}(\tau) = 0\ , \qquad e^{FS}(\tau) = \frac{(e\tau)_0}{\tau}\ , \tag{1.74}$$

where the integration constant $(e\tau)_0$ corresponds to the initial energy density per unit rapidity. Even if we consider a system, which is initially in equilibrium, where we can compute the above expressions explicitly, yielding

$$p_L^{FS}(\tau) = \frac{e(\tau_0)}{3} S\Big(\frac{\tau_0}{\tau}\Big)\ , \qquad S(x) = \frac{3}{2}\int_{-1}^{+1} d\cos(\theta) \frac{x^3 \cos^2(\theta)}{\sqrt{\sin^2(\theta) + x^2\cos^2(\theta)}} = \frac{3}{2}x^3\left[\frac{x}{x^2-1} + \frac{\arcsin(\sqrt{1-x^2})}{(1-x^2)^{3/2}}\right]\ , \tag{1.75}$$

$$e^{FS}(\tau) = e(\tau_0) C\Big(\frac{\tau_0}{\tau}\Big)\ , \qquad C(x) = \frac{1}{2}\int_{-1}^{+1} d\cos(\theta)\ x\ \sqrt{\sin^2(\theta) + x^2\cos^2(\theta)} = \frac{1}{2}x\left[x + \frac{\arcsin(\sqrt{1-x^2})}{(1-x^2)^{1/2}}\right]\ , \tag{1.76}$$

we find that for $\tau \gg \tau_0$, one has $p_L^{FS}(\tau)/e^{FS}(\tau) \sim \left(\frac{\tau_0}{\tau}\right)^2$ as anticipated, i.e. the longitudinal expansion drives the system out of equilibrium and towards the free-streaming attractor.

First interaction correction: Based on the free-streaming attractor solution, we can also compute the first interaction correction. By solving the Boltzmann equation to the next order in the interaction rate, we obtain the first interaction correction δf to the free-streaming attractor solution, according to

$$\partial_\tau \delta f = p_\parallel \partial_{p_\parallel} \delta f - \frac{1}{\tau_R}\left(f^{FS} - f^{FS}_{\rm eq}\right) \tag{1.77}$$

which can again be solved in closed form yielding

$$\delta f(\tau, p_T, p_\parallel) = -\int_{\tau_0}^{\tau} d\tilde{\tau} \frac{1}{\tau_R(\tilde{\tau})}\left(f^{FS}(\tilde{\tau}, p_T, \frac{\tau}{\tilde{\tau}}p_\parallel) - f^{FS}_{\rm eq}(\tilde{\tau}, p_T, \frac{\tau}{\tilde{\tau}}p_\parallel)\right) \tag{1.78}$$

Evaluating the first interaction correction to the longitudinal pressure is surprisingly simple, as considering

$$\delta p_L(\tau) = \nu_{\rm eff}\int_p \frac{p_\parallel^2}{\sqrt{p_T^2 + p_\parallel^2}}\delta f(\tau, p_T, p_\parallel) \tag{1.79}$$

only the term proportional to the equilibrium distribution $f^{FS}_{\rm eq}$ contributes, as the free-streaming attractor solution f_{FS} features a vanishing longitudinal pressure, such that

$$\delta p_L(\tau) = \int_{\tau_0}^{\tau} d\tilde{\tau} \frac{1}{\tau_R(\tilde{\tau})} \int_p \frac{p_\parallel^2}{\sqrt{p_T^2 + p_\parallel^2}} f^{FS}_{\rm eq}(\tilde{\tau}, p_T, \frac{\tau}{\tilde{\tau}}p_\parallel) \tag{1.80}$$

Evaluating, this according to (1.75), one finds

$$\delta p_L(\tau) = \int_{\tau_0}^{\tau} d\tilde{\tau} \frac{1}{\tau_R(\tilde{\tau})} \frac{e_{FS}(\tilde{\tau})}{3} S\Big(\frac{\tilde{\tau}}{\tau}\Big) \tag{1.81}$$

such that upon introducing suitable pre-factors an taking the limit $\tau_0 \to 0$, this can be expressed as

$$\delta p_L(\tau) = \frac{e_{FS}(\tau)}{3}\frac{\tau}{\tau_R(\tau)}\int_0^1 d\left(\frac{\tilde{\tau}}{\tau}\right) \underbrace{\frac{\tau_R(\tau)}{\tau_R(\tilde{\tau})}}_{=\left(\frac{\tau}{\tilde{\tau}}\right)^{1/4}} \underbrace{\frac{e_{FS}(\tilde{\tau})}{e_{FS}(\tau)}}_{=\left(\frac{\tau}{\tilde{\tau}}\right)} S\Big(\frac{\tilde{\tau}}{\tau}\Big) = \frac{e_{FS}(\tau)}{3}\frac{\tau}{\tau_R(\tau)}\gamma_S \tag{1.82}$$

with $\gamma_S = \int_0^1 dx\ x^{-5/4}\ S(x) \approx 0.441913$. By combining the leading order free-streaming and first interaction correction we thus conclude, that at early times the long. pressure of the system behaves as

$$p_L/e = 0 + \frac{\gamma_s}{3}\frac{\tau}{\tau_R(\tau)} + \cdots \tag{1.83}$$

we see that while the free-streaming attractor features a maximal anisotropy of the system, interactions drive the system towards a more isotropic state. Evidently, for $\tau/\tau_R \sim 1$ the early time expansion breaks down, and one needs to resort to a numerical solution of the problem as illustrated in Fig. XXX, smoothly interpolates between the early and late time behavior.

Summarising our discussion, we find that at early times the system features a dynamical attractor, which inevitably results in a close to zero long. pressure, and the system appears macroscopically as free-streaming regardless of the details of the initial conditions. Subsequently, interactions start to establish a finite longitudinal pressure in the system, which eventually approaches the behaviour expected from the hydrodynamic late time expansion, as can be seen from the numerical solutions to various kinetic theories shown in Fig. XXX. Generally, this happens at a stage, where the system is still quite far from isotropic, and exhibits an order one pressure anisotropy. However, even under these circumstances, viscous hydrodynamics is able to describe the subsequent approach towards equilibrium, which occurs on much larger time scales.

1.3.5 Dynamical attractors beyond Bjorken flow

Now to gain further insights into the concept of dynamical attractors, that emerge due to the competing effects of anisotropic expansions and interactions, we consider a situation where instead of Bjorken expansion, the system is subject to a periodic expansion/contraction of the space-time as expressed by the scale factor

$$a(\tau) = 1 + \delta a(\tau)\ , \qquad \delta a(\tau) = \alpha \sin(\Omega\tau)\theta(\tau)\ , \tag{1.84}$$

such that until $\tau = 0$, the system is in equilibrium and then starts to periodically expand and compress. Clearly, the advantage of this setup is that in comparison to usual Bjorken flow, we can independently control

- α magnitude of the perturbation
- Ω rate of expansion

Since the onset of the anisotropic expansion will drive the system out-of-equilibrium, the near equilibrium regime, where hydrodynamics is expected to provide an accurate description is then given by

$$\alpha \ll 1 \,, \qquad \Omega\tau_\pi \ll 1 \,, \tag{1.85}$$

which describes the repsonse to small perturbations on large time scales, compared to the intrinsic time scales of the system. We note that in this linear regime, the expansion rate of the system is then simply given by

$$\frac{\dot{a}}{a} \overset{\alpha\ll 1}{\simeq} \alpha\Omega\cos(\Omega\tau)\theta(\tau) \,, \tag{1.86}$$

Now to assess the validity of hydrodynamics, we will once again compare microscopic and macroscopic solutions in hydrodynamics and kinetic theory.

Navier Stokes Hydro

We again begin with the Navier Stokes solution, which is still given as in (1.47) by

$$\pi^\eta_{\ \eta}\Big|_{NS} = \frac{4}{3}\eta\frac{\dot{a}}{a} \,. \tag{1.87}$$

1.3.6 Isreal-Stuart Hydro

Now to describe the evolution in Israel-Stuart hydrodynamics, we need to solve the ordinary differential equation

$$\partial_\tau \pi^\eta_{\ \eta} = -\frac{1}{\tau_\pi}(\pi^\eta_{\ \eta} - \frac{4}{3}\eta\frac{\dot{a}}{a}) \tag{1.88}$$

Since in the regime of weak perturbations, the transport coefficient τ_π can be considered as constant, the evolution equation for the shear-stress tensor decouples from the evolution equation for the energy density e,

$$\partial_\tau e = -\frac{\dot{a}}{a}\left(e + p - \pi^\eta_{\ \eta}\right) \,. \tag{1.89}$$

While it is relatively straightforward to directly solve the ODE, it proves more insightful to employ the method of Green's functions to solve the equation. By inserting the explicit expression for the expansion rate and re-arranging the terms in the evolution equation according to

$$(1 + \tau\partial_\tau)\pi^\eta_{\ \eta} = \frac{4}{3}\eta\ \alpha\Omega\ \cos(\Omega t) \tag{1.90}$$

we can obtain the solution in terms of a retarded Green's function, as

$$\pi^\eta_{\ \eta}(\tau) = \int_0^\tau d\tau'\ G_R(\tau - \tau')\ \delta a(\tau') \tag{1.91}$$

where the retarded Green's function satisfies the evolution equation

$$(1 + \tau\partial_\tau)G_R(\tau - \tau') = \frac{4}{3}\eta\ \delta'(\tau - \tau') \tag{1.92}$$

we assumed that initially at $\tau = 0$ there is no shear-stress as the system is in equilibrium. Naturally, this is easily solved in Fourier space, where upon defining

$$G_R(\omega) = \int d\tau\ G_R(\tau - \tau')\ e^{i\omega(\tau-\tau')} \,, \qquad G_R(\tau - \tau') = \int \frac{d\omega}{(2\pi)}\ G_R(\omega)\ e^{-i\omega(\tau-\tau')} \tag{1.93}$$

the equation becomes

$$(1 - i\omega\tau_\pi)G_R(\omega) = -i\omega \tag{1.94}$$

such that we obtain

$$G_R(\omega) = \frac{4}{3}\eta\ \frac{-i\omega}{1 - i\omega\tau_\pi} \,. \tag{1.95}$$

Since in Fourier space, the convolution in (1.91) simply turns into a product, and we can determine the real-time response of the system as

$$\pi^{\eta}{}_{\eta}(\tau) = \int \frac{d\omega}{2\pi}\, G(\omega)\, \delta a(\omega)\, e^{-i\omega\tau} \tag{1.96}$$

where for the specific perturbation considered, we have

$$\delta a(\omega) = \frac{\alpha}{2}\left[\frac{1}{\omega+\Omega+i\epsilon} - \frac{1}{\omega-\Omega+i\epsilon}\right] \tag{1.97}$$

where the $i\epsilon$ prescription ensures the emergence of the Heavyside theta function $\theta(\tau)$ in (1.84). Now in order to compute the real-time response in (1.96) we can perform the integration using the residue theorem, and as usual close the integration contour through the upper half plane for $\tau > 0$ or lower half plane for $\tau < 0$. Since all singularities of the integrand are located in the lower half plane, we obtain $\pi^{\eta}{}_{\eta}(\tau < 0) = 0$, while for $\tau > 0$ we receive contributions from

- Singularity of the Green's function at $\omega = -i/\tau_\pi$
- Singularities of the metric perturbation at $\omega = \pm\Omega - i\epsilon$

signifying that the solution will feature a combination of exponentially decaying behavior and oscillatory behavior due to the periodic space-time expansion/contraction. Evaluating this explicitly, one finds

$$\pi^{\eta}{}_{\eta}(\tau) = \frac{4}{3}\eta\ \Omega\alpha\ \frac{\left[\cos(\Omega\tau) + \Omega\tau_\pi \sin(\Omega\tau) - e^{-\tau/\tau_\pi}\right]}{1+\Omega^2\tau_\pi^2}\theta(\tau) \tag{1.98}$$

Note that if instead of considering the system to be initially in equilibrium, we considered a finite initial shear-stress, we obtain an extra contribution

$$+\pi^{\eta}{}_{\eta}(\tau = 0)\ e^{-\tau/\tau_\pi} \tag{1.99}$$

which simply corresponds to the homogenous solution of the linear ODE.

Now to characterize the behavior of the solutions, we need to distinguish between two different regimes

- Early times ($t \lesssim \tau_\pi$): We observe the decay of transient excitations $\sim e^{-t/\tau_\pi}$, associated with the shear-stress. Such non-hydrodynamic modes are associated with singularities of the retarded Green's function, where in this case they emerge from the single pole of $G_R(\omega)$
- Late times ($t \gtrsim \tau_\pi$): Since transient excitations decay exponentially on a time scale $\sim \tau_\pi$, the evolution at late times is characterized by a universal time dependence for different initial conditions, which corresponds to the hydrodynamic attractor.

We conclude that in the long-time limit the evolution follows the

$$\text{Dynamical attractor:}\quad \pi^{\eta}{}_{\eta}(\tau \gg \tau_\pi) = \frac{4}{3}\eta\ \Omega\alpha\ \frac{[\cos(\Omega\tau) + \Omega\tau_\pi \sin(\Omega\tau)]}{1+\Omega^2\tau_\pi^2} \tag{1.100}$$

Now to obtain the strict hydrodynamic limit, we also need to require that $\Omega\tau_\pi \ll 1$, such that the perturbation is slow on the natural time scale of the system and the system remains close to equilibrium. In this case the attractor solution reduces

$$\pi^{\eta}{}_{\eta}(\tau \gg \tau_\pi) \overset{\Omega\tau_\pi \ll 1}{=} \pi^{\eta}{}_{\eta}(\tau \gg \tau_\pi) = \frac{4}{3}\eta\ \Omega\alpha\ \cos(\Omega\tau) = \pi^{\eta}{}_{\eta}(\tau)\|_{NS} \tag{1.101}$$

such that after the decay of transients, the system shows the expected Navier-Stokes behavior. However, the dynamical attractor solution emerges also for $\Omega\tau_\pi \gtrsim 1$, where it shows significant deviations from Navier-Stokes hydrodynamics. Specifically, for $\Omega\tau_\pi \gtrsim 1$, we may re-express

$$\pi^{\eta}{}_{\eta}(\tau \gg \tau_\pi) = \frac{4}{3}\eta\ \Omega\alpha\ \frac{\cos(\Omega\tau - \phi)}{\sqrt{1+\Omega^2\tau_\pi^2}}\,, \qquad \phi = \arctan(\Omega\tau_\pi) \tag{1.102}$$

the response of the system to the metric perturbation becomes weaker and experiences a phase-shift. Strikingly, in the limit $\Omega\tau_\pi \gg 1$, the system is unable to follow the rapidly varying perturbation and the response ceases to exist.

Summarizing our discussion, we have seen that the competition of expansion and interations leads to the emergence of dynamical attractors

- describe the long-time evolution of non-equilibrium systems
- emerge after the deacy of transients $\tau \gg \tau_\pi$, as described by the singularities of the retarded Green's function of the energy-momentum tensor (in the linear regime)

While in Israel-Stuart hydrodynamics the analytic structure of $G(\omega)$ is rather simple, with a single pole on the negative imaginary axis, the investigation of these structures in microscopic theories is an area of active investigation.

Generally speaking, causality and stability limits the possible singularities to be in the lower complex half plane, however the precise structure can be very different for different microscopic theories. If we consider the case of spatially homogenous perturbations, as discussed above,

- Isreal-Stuart / RTA Kinetic Theory : Single pole at $\omega = -i/\tau_\pi$
- Scalar kinetic theory: Non-analytic region on the negative imaginary axis, as determined by the eigenvalues of the linearized collision operator.
- Strongly coupled $\mathcal{N} = 4$ SYM (Holography): Infinite tower of complex conjugated pairs of poles, corresponding to the quasi-normal frequencies of holographic black holes.

1.4 Hydrodynamics with transverse expansion

We have seen that if we neglect the transverse expansion of the system, eventually a conformal system will always approach equilibrium, as in Bjorken flow the expansion rate $\frac{\dot{a}}{a} = 1/\tau$, decreases more rapidly than the interaction rate $\frac{\eta/s}{T(\tau)}$ which behaves as $\sim \tau^{-1/4}$ for a free-streaming system. However, this situation changes drastically, if we also account for the transverse expansion of the system, as in this case a small or weakly interacting system can begin to expand in all spatial directions and become very dilute prior to the onset of equilibration. Strikingly, this question on the applicability of hydrodynamics in small system is not only of theoretical interest, but also of enormous phenomenological relevance, as typical signatures of hydrodynamic behavior, such as anisotropic collective flow, have also been observed in small systems realized in the collisions of light nuclei, such as $O^{16} + O^{16}$ or with some restrictions even $p + Pb^{208}$ or $p + p$ collisions.