Relativistic Chapman–Enskog Expansion

From the Boltzmann Equation to First-Order Dissipative Hydrodynamics

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Purpose of the lecture

This lecture explains how the relativistic Chapman–Enskog expansion derives first-order dissipative hydrodynamics from the relativistic Boltzmann equation.

The central kinetic object is the one-particle distribution function

$$f_k(x) \equiv f(x,k),$$

while the hydrodynamic variables are only a few moments of this function. The Chapman-Enskog method constructs a controlled approximation to f_k when microscopic and macroscopic scales are well separated.

The final first-order result is the relativistic Navier–Stokes equation with constitutive equations

$$\Pi = -\zeta\theta, \qquad n^\mu = \kappa\nabla^\mu\alpha_0, \qquad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}.$$

The derivation is microscopic because the coefficients ζ, κ, and η are expressed in terms of collision integrals.

Historical background

A first important step was taken by Hilbert that introduced a perturbative approach to the Boltzmann equation. Hilbert introduced the idea of expanding the distribution function around local equilibrium using a small parameter related to the separation between microscopic and macroscopic scales.
This made clear that hydrodynamics is not simply an assumption, but a limiting description obtained when collisions drive the system rapidly toward local equilibrium.

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The Chapman–Enskog expansion is the traditional method for deriving fluid dynamics from the Boltzmann equation. It was originally developed for non-relativistic dilute gases, but Israel showed that the same formalism can be extended to relativistic kinetic theory with only minor modifications.

Historical background

In modern language, the method is the microscopic realization of the gradient expansion. It assumes that the single-particle distribution function depends only on the five fluid-dynamical variables,

$$T, \qquad \mu, \qquad u^\mu,$$

and on their gradients $f_k(x) = f_k\big(n(x), T(x), u^\mu(x); \partial n, \partial T, \partial u, \dots\big)$.

The non-equilibrium correction is then organized as an expansion in powers of a small parameter ϵ that will be related to the Knudsen (Kn) number,

$$f_k = f_k^{(0)} + \epsilon f_k^{(1)} + \epsilon^2 f_k^{(2)} + \cdots .$$

Orders of the expansion and limitations

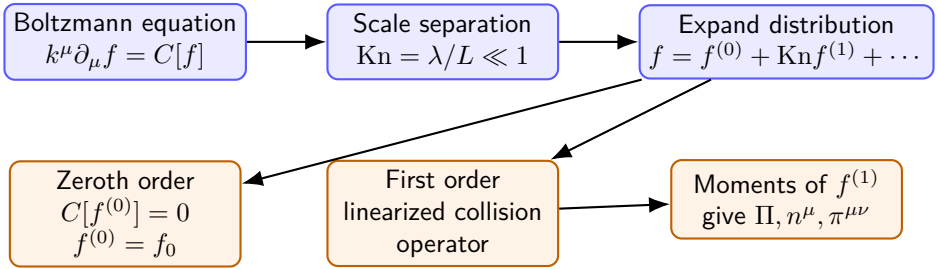
At zeroth order the Chapman–Enskog expansion gives local equilibrium and ideal hydrodynamics. At first order it gives Navier–Stokes theory:

$$\Pi = -\zeta\theta, \qquad n^\mu = \kappa\nabla^\mu\alpha_0, \qquad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}.$$

Keeping higher orders gives the Burnett and super-Burnett equations. In principle, the solution can be constructed to arbitrarily high order in the Knudsen number.

However, as first emphasized by Grad, the Chapman–Enskog series is asymptotic. Moreover, in the relativistic case the resulting equations of motion are unstable and therefore not useful as realistic dynamical equations beyond their asymptotic meaning.
Despite this limitation, Chapman–Enskog remains fundamental because it reveals the asymptotic hydrodynamic behavior of the Boltzmann equation and provides microscopic expressions for transport coefficients.

Diagram: logic of Chapman–Enskog



The method converts a microscopic kinetic equation into macroscopic constitutive relations.

Conventions and notation

The particle momentum is on shell,

$$k^\mu = (k^0, \mathbf{k}), \qquad k^0 = \sqrt{\mathbf{k}^2 + m^2},$$

and the fluid four-velocity satisfies

$$u^\mu u_\mu = 1.$$

The local-rest-frame particle energy is

$$E_k \equiv k^\mu u_\mu.$$

The spatial projector is

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

All irreducible tensor structures are defined in relation to u^μ.
For a vector $A^{\langle\mu\rangle} = \Delta^\mu{}_\nu A^\nu$ for a rank-two tensor $A^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$
where $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$.
Thus $A^{\langle\mu\nu\rangle}$ is symmetric, and traceless.

Relativistic Boltzmann equation

The microscopic equation is

$$k^\mu \partial_\mu f_k = C[f].$$

For binary elastic collisions,

$$C[f] = \frac{1}{\nu} \int \mathrm{d}K' \, \mathrm{d}P \, \mathrm{d}P' \, W_{kk' \to pp'} \left(f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} - f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} \right),$$

with

$$\tilde{f}_k = 1 - a f_k, \qquad a = 1, -1, 0$$

for fermions ($a = 1$), bosons ($a = -1$), and a classical gas ($a = 0$). While $\nu = 2$ is the symmetry factor for identical particles.

The transition rate $W_{kk' \to pp'}$ is Lorentz invariant. While $dK \equiv d^3k/[(2\pi)^3 k^0]$ is the Lorentz invariant momentum space volume.

Kinetic moments and hydrodynamic currents

For notational convenience, we introduce the following momentum average:

$$\langle \cdots \rangle \equiv \int \mathrm{d}K \, (\cdots) f_k.$$

The conserved current and energy-momentum tensor can be written as

$$N^\mu = \int \mathrm{d}K \, k^\mu f_k = \langle k^\mu \rangle \,, \qquad T^{\mu\nu} = \int \mathrm{d}K \, k^\mu k^\nu f_k = \langle k^\mu k^\nu \rangle \,.$$

We have seen that using the Boltzmann equation gives

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0.$$

These equations govern only the conserved fields. They do not determine all higher moments of f_k.

How do we express dissipative moments in terms of hydrodynamic variables?

Chapman–Enskog answers this by assuming that deviations from local equilibrium are gradient corrections controlled by the Knudsen number.

Landau-frame decomposition

The Landau velocity is defined by

$$T^{\mu\nu}u_\nu = \varepsilon u^\mu.$$

The particle current is decomposed as

$$N^\mu = nu^\mu + n^\mu, \qquad u_\mu n^\mu = 0.$$

The energy-momentum tensor is decomposed as

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu},$$

with

$$u_\mu \pi^{\mu\nu} = 0, \qquad \pi^\mu{}_\mu = 0.$$

In the Landau frame there is no energy diffusion. The remaining vector dissipative current is particle or charge diffusion.

Local equilibrium and matching

The local-equilibrium distribution is

$$f_{0k} = \left[\exp(\beta_0 E_k - \alpha_0) + a\right]^{-1}, \qquad \beta_0 = 1/T, \qquad \alpha_0 = \mu/T.$$

The parameters α_0 and β_0 are defined by the matching conditions[1]:

$$n \equiv n_0 = \int \mathrm{d}K\, k^\mu u_\mu\, f_{0k} = \langle E_k \rangle_0\,, \qquad \varepsilon \equiv \varepsilon_0 = \int \mathrm{d}K \left(k^\mu u_\mu\right)^2 f_{0k} = \left\langle E_k^2 \right\rangle_0.$$

The pressure split is

$$P_0 = -\frac{1}{3}\left\langle \Delta^{\mu\nu} k_\mu k_\nu \right\rangle_0, \qquad \Pi = -\frac{1}{3}\left\langle \Delta^{\mu\nu} k_\mu k_\nu \right\rangle_\delta \qquad where \qquad \langle \cdots \rangle_\delta = \langle \cdots \rangle - \langle \cdots \rangle_0$$

The matching conditions fix the local temperature, chemical potential, and fluid velocity inside the nonequilibrium distribution.

[1] Once dissipative currents are introduced, the fluid is no longer in local thermodynamic equilibrium. Therefore, equilibrium quantities such as $\alpha_0, \beta_0, s_0, P_0$ must be defined through a fictitious local-equilibrium state. This fictitious state is chosen so that its particle density and energy density coincide with the local-rest-frame densities.

Derivative decomposition

Decompose the derivative into a part parallel and a part orthogonal to the flow:

$$\partial_\mu = u_\mu D + \nabla_\mu, \qquad D \equiv u^\nu \partial_\nu, \qquad \nabla_\mu \equiv \Delta_\mu{}^\nu \partial_\nu.$$

Then the left-hand side of the Boltzmann equations becomes

$$k^\mu \partial_\mu f_k = E_k D f_k + k^\mu \nabla_\mu f_k,$$

and the Boltzmann equation becomes

$$D f_k + \frac{1}{E_k} k^\mu \nabla_\mu f_k = \frac{1}{E_k} C[f].$$

This equation separates comoving time variation from spatial gradients in a covariant way.

Knudsen scaling

In the local rest frame, ∇_μ defines an inverse macroscopic length scale L^{-1}, while D defines the inverse macroscopic time scale $\bar{\tau}^{-1}$. Let me introduce

$$\nabla_\mu = L^{-1}\hat{\nabla}_\mu, \qquad D = \bar{\tau}^{-1}\hat{D},$$

where $\hat{\nabla}_\mu$ and $\hat{D}$ are dimensionless derivatives.

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Multiplying the Boltzmann equation by the microscopic mean free path λ gives

$$\mathrm{Kn}_t \hat{D} f_k + \frac{\mathrm{Kn}_s}{E_k} k^\mu \hat{\nabla}_\mu f_k = \frac{\lambda}{E_k} C[f],$$

with

$$\mathrm{Kn}_s = \frac{\lambda}{L}, \qquad \mathrm{Kn}_t = \frac{\lambda}{\bar{\tau}}.$$

In the Chapman–Enskog construction one assumes $\mathrm{Kn}_t = \mathrm{Kn}_s \equiv \mathrm{Kn}$. This will permit to replace space gradients with time gradients and rewrite the solution f_k in terms of powers of space gradients.

Chapman–Enskog expansion

The distribution function is expanded as

$$f_k = f_k^{(0)} + \mathrm{Kn} f_k^{(1)} + \mathrm{Kn}^2 f_k^{(2)} + \cdots .$$

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The collision term is expanded consistently:

$$C[f] = C^{(0)} + \mathrm{Kn} C^{(1)} + \mathrm{Kn}^2 C^{(2)} + \cdots .$$

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Equating equal powers of Kn gives

$$\begin{aligned} 0 &= C^{(0)}, \\ [\hat{D} f_k]^{(1)} + \frac{1}{E_k} k^\mu \hat{\nabla}_\mu f_k^{(0)} &= \frac{\lambda}{E_k} C^{(1)}, \\ [\hat{D} f_k]^{(2)} + \frac{1}{E_k} k^\mu \hat{\nabla}_\mu f_k^{(1)} &= \frac{\lambda}{E_k} C^{(2)}. \\ &\cdots \end{aligned}$$

The hierarchy is recursive. Once $f^{(0)}$ is known, the first-order equation determines $f^{(1)}$, up to collision invariants fixed by matching.

Notice that $[\hat{D} f_k]^{(n)} \neq \hat{D} f_k^{(n)}$

A subtle but essential point is that

$$[\hat{D} f_k]^{(n)} \neq \hat{D} f_k^{(n)}$$

in general.
The reason is that comoving derivatives are eliminated using the hydrodynamic equations, and those equations themselves contain gradient corrections. Schematically,

$$\hat{D} f_k^{(n)} = [\hat{D} f_k^{(n)}]^{(1)} + \mathrm{Kn}[\hat{D} f_k^{(n)}]^{(2)} + \cdots .$$

Chapman–Enskog is therefore not a naive expansion of f_k alone because it is a combined expansion of the kinetic equation and the hydrodynamic evolution.

Zeroth order: local equilibrium / Ideal Hydro

Let us solve the first equation: the zeroth-order equation is

$$C^{(0)} = 0.$$

We have seen that the solution is

$$f_k^{(0)} = f_{0k} = \left[\exp(\beta_0 E_k - \alpha_0) + a\right]^{-1}.$$

where $\alpha_0(x)$,$\beta_0(x)$ and $u^\mu(x)$ depend on space-time.

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where $\alpha_0(x)$,$\beta_0(x)$ and $u^\mu(x)$ depend on space-time.
At this order, by isotropy of f_{0k} in the local rest frame,

$$\Pi^{(0)} = -\frac{1}{3}\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_0 - P_0 = 0, \qquad n^{\mu(0)} = \left\langle k^{\langle\mu\rangle} \right\rangle_0 = 0, \qquad \pi^{\mu\nu(0)} = \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle_0 = 0.$$

Thus the conservation laws reduce to ideal relativistic hydrodynamics: $\partial_\mu N^\mu_{(0)} = 0$ and $\partial_\mu T^{\mu\nu}_{(0)} = 0$.
Local equilibrium makes the collision term vanish, but it does not solve the full Boltzmann equation unless α_0, β_0, and u^μ are spacetime independent.
Ideal Hydro can be interpreted as the zeroth-order truncation of an expansion in terms of Kn

Matching conditions on corrections

The matching conditions must be satisfied for all $n \geq 1$, therefore the corrections obey

$$\int \mathrm{d}K\, E_k f_k^{(n)} = 0, \qquad \int \mathrm{d}K\, E_k^2 f_k^{(n)} = 0, \qquad \int \mathrm{d}K\, E_k k^{\langle\mu\rangle} f_k^{(n)} = 0^2.$$

These conditions mean that $f^{(n)}$ does not shift the matched values of

$$n_0, \qquad \varepsilon_0, \qquad u^\mu.$$

Physically, they keep the definition of the local equilibrium state fixed. This guarantee that to any order of approximation the solution depends on α_0, β_0 and u^μ

[2] This Eq. is obtained by the following projection $\Delta^\alpha_\mu T^{\mu\nu} u_\nu = \epsilon u^\mu \Delta^\alpha_\mu = 0$. This guarantee that there is no flow of energy relative to u^μ.

First-order equation

Now we solve the first-order equation. We construct the solution of Chapmann-Enskog expansion to first order in Knudsen number.

Using $f_k^{(0)} = f_{0k}$, the first-order equation is

$$[\hat{D} f_{0k}]^{(1)} + \frac{1}{E_k} k^\mu \hat{\nabla}_\mu f_{0k} = -\lambda \hat{C} f_k^{(1)}.$$

The linearized collision operator is

$$\hat{C} f_k^{(1)} = \frac{1}{\nu E_k} \int \mathrm{d}K' \, \mathrm{d}P \, \mathrm{d}P' \, W_{kk' \to pp'} f_{0k} f_{0k'} \tilde{f}_{0p} \tilde{f}_{0p'} \left(\frac{f_k^{(1)}}{f_{0k} \tilde{f}_{0k}} + \frac{f_{k'}^{(1)}}{f_{0k'} \tilde{f}_{0k'}} - \frac{f_p^{(1)}}{f_{0p} \tilde{f}_{0p}} - \frac{f_{p'}^{(1)}}{f_{0p'} \tilde{f}_{0p'}} \right).$$

The left-hand side is the gradient source; the right-hand side is the collision integral linearized around local equilibrium.

Exercise: verify this relation.

First-order equation: Derivatives of f_{0k}

In the left-hand side of the Boltzmann equation we need to calculate $\hat{D}f_{0k}$ and $\hat{\nabla}_\mu f_{0k}$.
From $f_{0k} = \left[e^{\beta_0 E_k - \alpha_0} + a\right]^{-1}$ we have that space-time dependence are in $\alpha_0(x)$,$\beta_0(x)$ and $u^\nu(x)$ therefore, we obtain

$$\hat{D}f_{0k} = -f_{0k}\tilde{f}_{0k}\left(E_k\hat{D}\beta_0 + \beta_0 k^{\langle\nu\rangle}\hat{D}u_\nu - \hat{D}\alpha_0\right),$$

$$\hat{\nabla}_\mu f_{0k} = -f_{0k}\tilde{f}_{0k}\left(E_k\hat{\nabla}_\mu\beta_0 + \beta_0 k^{\langle\nu\rangle}\hat{\nabla}_\mu u_\nu - \hat{\nabla}_\mu\alpha_0\right).$$

where we have used that $u^\nu\hat{D}u_\nu = u^\nu\hat{\nabla}_\mu u_\nu = 0$.

The gradients of α_0, β_0, and u^μ are the thermodynamic forces that deform the local-equilibrium distribution.

Eliminating time derivatives

To first order in Kn, conservation laws give

$$\hat{D}\alpha_0 = \frac{(\varepsilon_0 + P_0)J_{20} - n_0 J_{30}}{D_{20}}\hat{\theta} + \mathcal{O}(\mathrm{Kn}^2),$$

$$\hat{D}\beta_0 = \frac{(\varepsilon_0 + P_0)J_{10} - n_0 J_{20}}{D_{20}}\hat{\theta} + \mathcal{O}(\mathrm{Kn}^2),$$

$$\hat{D}u^\mu = \frac{1}{\varepsilon_0 + P_0}\hat{\nabla}^\mu P_0 + \mathcal{O}(\mathrm{Kn}^2).$$

Exercise: verify the above relations.

Pay attention! This is the key point of Chapman–Enskog theory: time-like derivatives are not kept as independent structures, but rewritten in terms of spatial gradients order by order. In the expression above, we have replaced all time-like derivatives by space-like gradients using the conservation equation below

$$\hat{D}n + n\hat{\theta} + \hat{\nabla}_\mu n^\mu - n^\mu \hat{D}u_\mu = 0, \tag{3.45}$$

$$\hat{D}\varepsilon + (\varepsilon + P_0 + \Pi)\hat{\theta} - \pi^{\mu\nu}\hat{\sigma}_{\mu\nu} = 0, \tag{3.46}$$

$$(\varepsilon + P_0 + \Pi)\hat{D}u^\alpha - \hat{\nabla}^\alpha(P_0 + \Pi) - \pi^{\alpha\beta}\hat{D}u_\beta + \Delta^\alpha_{\ \nu}\hat{\nabla}_\mu \pi^{\mu\nu} = 0. \tag{3.47}$$

Thermodynamic integrals

where we have defined

$$I_{nq} = \frac{(-1)^q}{(2q+1)!!} \int \mathrm{d}K \, f_{0k} E_k^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q ,$$

$$J_{nq} = \frac{(-1)^q}{(2q+1)!!} \int \mathrm{d}K \, f_{0k} \tilde{f}_{0k} E_k^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q .$$

They satisfy

$$n_0 = I_{10}, \qquad \varepsilon_0 = I_{20}, \qquad P_0 = I_{21},$$

and

$$dI_{nq} = J_{nq} d\alpha_0 - J_{n+1,q} d\beta_0 .$$

These integrals contain equilibrium thermodynamics. They are distinct from the collision matrices, which contain the microscopic scattering kernel.

The first-order source

After substituting the first-order time-derivative relations, the equation becomes

$$f_{0k}\tilde{f}_{0k}\left(A_k\hat{\theta}+B_k k^{\langle\mu\rangle}\hat{\nabla}_\mu\alpha_0+\frac{\beta_0}{E_k}k^{\langle\mu}k^{\nu\rangle}\hat{\sigma}_{\mu\nu}\right)=\lambda\hat{C}f_k^{(1)}.$$

Here

$$A_k=\frac{(\varepsilon_0+P_0)J_{10}-n_0J_{20}}{D_{20}}E_k-\frac{(\varepsilon_0+P_0)J_{20}-n_0J_{30}}{D_{20}}+\frac{\beta_0}{3E_k}\Delta^{\alpha\beta}k_\alpha k_\beta,$$

$$B_k=h_0^{-1}-\frac{1}{E_k},\qquad h_0=\frac{\varepsilon_0+P_0}{n_0}.$$

The source splits into scalar, vector, and rank-two tensor sectors. This is the structural reason why bulk, diffusion, and shear transport can be treated separately.

Zero modes of $\hat{C}$

The linearized collision operator has five zero modes:

$$\hat{C}(f_{0k}\tilde{f}_{0k}) = 0, \qquad \hat{C}(f_{0k}\tilde{f}_{0k}E_k) = 0,$$

$$\hat{C}(f_{0k}\tilde{f}_{0k}k^{\langle\mu\rangle}) = 0.$$

These correspond to the collision invariants

$$1, \qquad k^\mu.$$

The operator can be inverted only outside its null space. Matching conditions fix the components of $f_k^{(1)}$ lying along these zero modes.

General first-order solution

Because the first-order equation is linear and its source is already decomposed into irreducible tensors, the solution has the form

$$\boxed{\frac{f_k^{(1)}}{f_{0k}\tilde{f}_{0k}} = \varphi_k^s \hat{\theta} + \varphi_k^v \beta_0 k^{\langle\mu\rangle} \hat{\nabla}_\mu \alpha_0 + \varphi_k^t \beta_0^2 k^{\langle\mu} k^{\nu\rangle} \hat{\sigma}_{\mu\nu} + \varphi_k^{\mathrm{hom}}.}$$

The homogeneous part is

$$\varphi_k^{\mathrm{hom}} = a_0 + a_1 E_k + a_{2\mu} k^{\langle\mu\rangle}.$$

The functions φ_k^s, φ_k^v, and φ_k^t depend only on E_k. The tensorial dependence is fixed by covariance and local isotropy.

Source moments

Define the projected source moments

$$\alpha_r^s = \int \mathrm{d}K\, E_k^r A_k f_{0k} \tilde{f}_{0k}$$

$$= \frac{(\varepsilon_0 + P_0)J_{10} - n_0 J_{20}}{D_{20}} J_{r+1,0} - \frac{(\varepsilon_0 + P_0)J_{20} - n_0 J_{30}}{D_{20}} J_{r0} - \beta_0 J_{r+1,1},$$

$$\alpha_r^v = \frac{1}{3}\int \mathrm{d}K\, E_k^r B_k (\Delta^{\alpha\beta} k_\alpha k_\beta) f_{0k} \tilde{f}_{0k} = J_{r+1,1} - h_0^{-1} J_{r+2,1},$$

$$\alpha_r^t = \frac{2}{15}\beta_0 \int \mathrm{d}K\, E_k^{r-1} (\Delta^{\alpha\beta} k_\alpha k_\beta)^2 f_{0k} \tilde{f}_{0k} = 2\beta_0 J_{r+3,2}.$$

These are the scalar amplitudes of the three thermodynamic driving forces.

Integral equations for the three channels

Projecting the first-order equation gives three independent integral equations:

$$\alpha_r^s = \frac{\lambda}{\nu}\int \mathrm{d}K\mathrm{d}K'\mathrm{d}P\mathrm{d}P'\, W f_{0k} f_{0k'} \tilde{f}_{0p}\tilde{f}_{0p'} E_k^{r-1}(\varphi_k^s + \varphi_{k'}^s - \varphi_p^s - \varphi_{p'}^s),$$

$$\alpha_r^v = \frac{\beta_0\lambda}{3\nu}\int \mathrm{d}K\mathrm{d}K'\mathrm{d}P\mathrm{d}P'\, W f_{0k} f_{0k'} \tilde{f}_{0p}\tilde{f}_{0p'} E_k^{r-1} k^{\langle\mu\rangle}(\varphi_k^v k_{\langle\mu\rangle} + \cdots),$$

$$\alpha_r^t = \frac{\beta_0^2\lambda}{5\nu}\int \mathrm{d}K\mathrm{d}K'\mathrm{d}P\mathrm{d}P'\, W f_{0k} f_{0k'} \tilde{f}_{0p}\tilde{f}_{0p'} E_k^{r-1} k^{\langle\mu}k^{\nu\rangle}(\varphi_k^t k_{\langle\mu}k_{\nu\rangle} + \cdots).$$

Polynomial expansion in energy

The unknown scalar functions are expanded as

$$\varphi_k^i = \sum_{n=0}^{N_i} \epsilon_n^i E_k^n, \qquad i = s, v, t.$$

Then the projected integral equations become algebraic systems:

$$\alpha_r^i = \lambda \sum_{n=0}^{N_i} A_{rn}^i \epsilon_n^i, \qquad i = s, v, t.$$

The matrices A_{rn}^i are collision matrices. They contain all dependence on the transition rate $W_{kk' \to pp'}$. Increasing N_i improves the approximation to the inverse collision operator, but in practice the series must be truncated.

Minimal truncation

The minimal nontrivial truncation is

$$N_s = 2, \qquad N_v = 1, \qquad N_t = 0.$$

After removing rows and columns associated with collision invariants, one obtains

$$\epsilon_2^s = \frac{\alpha_0^s}{\lambda A_{02}^s}, \qquad \epsilon_1^v = \frac{\alpha_0^v}{\lambda A_{01}^v}, \qquad \epsilon_0^t = \frac{\alpha_0^t}{\lambda A_{00}^t}.$$

The remaining coefficients are fixed by matching:

$$\begin{pmatrix} J_{10} & J_{20} \\ J_{20} & J_{30} \end{pmatrix} \begin{pmatrix} \epsilon_0^s \\ \epsilon_1^s \end{pmatrix} = - \begin{pmatrix} J_{30} \\ J_{40} \end{pmatrix} \epsilon_2^s,$$

$$J_{31}\epsilon_0^v = -J_{41}\epsilon_1^v.$$

Solutions of the matching constraints

Using the thermodynamic functions

$$G_{nm} = J_{n0}J_{m0} - J_{n-1,0}J_{m+1,0}, \qquad D_{nq} = J_{n+1,q}J_{n-1,q} - J_{nq}^2,$$

one obtains

$$\epsilon_0^s = \frac{D_{30}}{D_{20}} \frac{\alpha_0^s}{\lambda A_{02}^s},$$

$$\epsilon_1^s = \frac{G_{23}}{D_{20}} \frac{\alpha_0^s}{\lambda A_{02}^s},$$

$$\epsilon_0^v = -\frac{J_{41}}{J_{31}} \frac{\alpha_0^v}{\lambda A_{01}^v}.$$

These coefficients are precisely the homogeneous pieces needed to enforce the Landau matching conditions.

First-order bulk pressure

Once we have the complete expression for $f_k^{(1)}$ we can calculate the bulk pressure:

$$\Pi^{(1)} = -\frac{1}{3}\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle^{(1)} = -\frac{1}{3}\int \mathrm{d}K\, (\Delta^{\mu\nu} k_\mu k_\nu) f_k^{(1)} \tag{1}$$

$$= \left(\frac{J_{21} D_{30} + J_{31} G_{23}}{D_{20}} + J_{41} \right) \frac{\alpha_0^s}{\lambda A_{02}^s} \hat{\theta}. \tag{2}$$

Since $\mathrm{Kn} = \lambda / L$ and $\hat{\theta} = L\theta$, this becomes

$$\Pi = -\zeta \theta,$$

where

$$\boxed{\zeta = -\left(\frac{J_{21} D_{30} + J_{31} G_{23}}{D_{20}} + J_{41} \right) \frac{\alpha_0^s}{A_{02}^s}.}$$

Bulk viscosity is the scalar response to expansion or compression.

First-order diffusion current

The diffusion 4-current is

$$n^{\mu(1)} = \left\langle k^{\langle\mu\rangle} \right\rangle^{(1)} = \int \mathrm{d}K\, k^{\langle\mu\rangle} f_k^{(1)} \tag{3}$$

$$= \frac{D_{31}}{J_{31}} \frac{\beta_0 \alpha_0^v}{\lambda A_{01}^v} \hat{\nabla}^\mu \alpha_0. \tag{4}$$

Restoring dimensional gradients gives

$$n^\mu = \kappa \nabla^\mu \alpha_0,$$

with

$$\boxed{\kappa = \frac{D_{31}}{J_{31}} \frac{\beta_0 \alpha_0^v}{A_{01}^v}.}$$

First-order shear-stress tensor

The shear-stress tensor is

$$\pi^{\mu\nu(1)} = \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle^{(1)} = \int \mathrm{d}K\, k^{\langle\mu} k^{\nu\rangle} f_k^{(1)} \tag{5}$$

$$= 2J_{42} \frac{\beta_0^2 \alpha_0^t}{\lambda A_{00}^t} \hat{\sigma}^{\mu\nu}. \tag{6}$$

Restoring dimensional gradients gives

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$

with

$$\boxed{\eta = J_{42} \frac{\beta_0^2 \alpha_0^t}{A_{00}^t}.}$$

Shear viscosity is the tensorial response to anisotropic expansion of the local fluid element.

What has been derived?

The first-order Chapman–Enskog expansion gives

$$f_k = f_{0k} + \mathrm{Kn} f_k^{(1)} + \mathcal{O}(\mathrm{Kn}^2).$$

Taking the relevant moments yields

$$\Pi = -\zeta\theta, \qquad n^\mu = \kappa\nabla^\mu\alpha_0, \qquad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}.$$

Thus relativistic Navier–Stokes theory is the first-order truncation of the microscopic gradient expansion of the Boltzmann equation.
In Navier–Stokes dissipative currents are not evolved independently, but are instantaneously related to gradients.

Limitations of first-order relativistic Chapman–Enskog

The first-order result is important but limited.

- The Chapman–Enskog expansion is asymptotic.
- First-order relativistic Navier–Stokes equations are acausal in the standard formulation.
- They are unstable around equilibrium in the usual relativistic setting.
- Dissipative currents respond instantaneously to gradients.
- Relaxation times do not appear at first order.

This motivates transient theories such as Israel–Stewart-type approaches and modern moment methods, where Π, n^μ, and $\pi^{\mu\nu}$ become dynamical variables.

Backup slides

Massless limit

For a massless gas,

$$I_{n0} = 3I_{n1}, \qquad J_{n0} = 3J_{n1}.$$

Using these relations, the scalar source vanishes:

$$\alpha_r^s = 0.$$

Therefore

$$\zeta = 0.$$

In a conformal system, isotropic expansion cannot produce an independent bulk-viscous stress because there is no intrinsic microscopic scale other than temperature. Shear viscosity can still be nonzero, because it measures anisotropic momentum transport.

References and source material


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Recap: viscous hydrodynamic equations of motion

We have seen that: the exact conservation laws,

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0,$$

become hydrodynamic equations after decomposing

$$N^\mu = nu^\mu + n^\mu,$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu}.$$

Projecting along and orthogonally to u^μ, one obtains

$$Dn + n\theta + \nabla_\mu n^\mu - n^\mu Du_\mu = 0, \tag{3.45}$$

$$D\varepsilon + (\varepsilon + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{3.46}$$

$$(\varepsilon + P_0 + \Pi)Du^\alpha - \nabla^\alpha(P_0 + \Pi) - \pi^{\alpha\beta}Du_\beta + \Delta^\alpha{}_\nu \nabla_\mu \pi^{\mu\nu} = 0. \tag{3.47}$$

Notice that these equations evolve the primary hydrodynamic fields n, ε, and u^μ, but they are not closed until constitutive relations or relaxation equations are supplied for Π, n^μ, and $\pi^{\mu\nu}$.

Why does bulk viscosity appear with a minus sign?

The bulk-viscous pressure enters the stress tensor as

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu}.$$

Thus the effective isotropic pressure is

$$P_{\text{eff}} = P_0 + \Pi.$$

For an expanding fluid,

$$\theta \equiv \partial_\mu u^\mu > 0,$$

bulk viscosity must reduce the effective pressure in order to oppose the expansion. Therefore one needs

$$\Pi < 0 \qquad \text{when} \qquad \theta > 0.$$

This is implemented by the Navier–Stokes constitutive relation $\Pi = -\zeta\theta,\ \zeta > 0$.
The sign is fixed by dissipation: the entropy-production term contains

$$-\frac{\Pi\theta}{T} = \frac{\zeta\theta^2}{T} \geq 0.$$

Hence ζ is positive and the minus sign ensures that bulk viscosity opposes isotropic expansion or compression.

Why does bulk viscosity come only from the scalar channel?

At first order, the Chapman–Enskog correction decomposes into irreducible scalar, vector, and tensor parts:

$$\frac{f_k^{(1)}}{f_{0k}\tilde{f}_{0k}} = \phi_k^s \theta + \phi_k^v \beta_0 k^{\langle\mu\rangle} \nabla_\mu \alpha_0 + \phi_k^t \beta_0^2 k^{\langle\mu} k^{\nu\rangle} \sigma_{\mu\nu}.$$

The bulk pressure is a scalar moment:

$$\Pi = -\frac{1}{3} \int dK \, \Delta^{\mu\nu} k_\mu k_\nu \, f_k^{(1)}.$$

Since f_{0k} is isotropic in the local rest frame, irreducible tensor sectors are orthogonal. Therefore:

$$\text{scalar moment} \times \text{vector part} = 0,$$

$$\text{scalar moment} \times \text{traceless tensor part} = 0.$$

Thus only the scalar part proportional to θ contributes to Π.
In the local rest frame this is just angular orthogonality:

$$\ell = 0 \text{ bulk}, \qquad \ell = 1 \text{ diffusion}, \qquad \ell = 2 \text{ shear}.$$

Different angular sectors do not mix in isotropic momentum integrals.