BIP, Timisoara May 2026

L. REZZOLLA

Lecture Ia: 3+1 split in general relativity

Lecture Ib: ADM formulation

Lecture IIa: Relativistic Hydrodynamics

Lecture IIb: Numerical Methods for Hyperbolic PDEs

Lecture IIIa: Application to BNSs: bulk dynamics & GWs

Lecture IIIb: Application to BNSs: EM emission and neutrinos

Consider perfect fluid characterised by the energy-momentum tensor of the type

$$\boxed{T^{\mu\nu} = (e+p)u^{\mu}u^{\nu} + p\, g^{\mu\nu} + T^{\mu\nu}_{EM}} \qquad (1)$$

with a (rest-) mass current

$$\boxed{J^{\mu} = \rho\, u^{\mu}} \qquad (2)$$

where

$\underline{u}$: the fluid four-velocity and is a unit timelike vector, i.e., $\underline{u}\cdot\underline{u} = -1$;

$e := \rho(1+\epsilon)$: energy density of fluid part

ϵ : specific internal energy

ρ : rest-mass density

$T^{\mu\nu}_{EM} := F^{\mu}{}_{\lambda}F^{\lambda\nu} - \frac{1}{4}(F^{\alpha\beta}F_{\alpha\beta})g^{\mu\nu}$

: energy-momentum associated with the electromagnetic fields

$F^{\mu\nu}$: Faraday tensor

We next have to obtain a formulation of the equations of general-relativistic hydrodynamics (GRHD) and general-relativistic magnetohydrodynamics (GRMHD) expressing conservation of (rest) mass, energy and momentum

$$\begin{cases} \nabla_\mu J^\mu = 0 \\ u_\beta \nabla_\alpha T^{\alpha\beta} = 0 \\ h^\mu{}_\beta \nabla_\alpha T^{\alpha\beta} = 0 \\ \left.\begin{array}{l} \nabla_\mu F^{\mu\nu} = J^\nu \\ \nabla_\mu {}^*F^{\mu\nu} = 0 \end{array}\right\} \text{Maxwell equations} \; ; \quad {}^*F^{\mu\nu} := \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} = \sqrt{-g}\, \eta^{\alpha\beta\mu\nu} F_{\alpha\beta} \end{cases} \qquad (3)$$

in a formulation that is suitable for a numerical integration. This is because while all formulation are equivalent in a continuum limit, they are not when employed in a discretised representation.

The set of GRMHD equations (3), together with an equation of state $p = p(\rho, \epsilon)$, describe the evolution of a fluid (plasma) in terms of nine variables: $\rho, v^i, p, \epsilon, B^i$; hereafter $B^i = 0$ (GRHD).

Hyperbolic PDEs

From a mathematical point of view, the equations of relativistic hydrodynamics or magnetohydrodynamics are "nonlinear hyperbolic" equations. You will find a detailed discussion in the extra material of this lecture but before discussing nonlinearity, let's start simple with linear hyperbolic PDEs

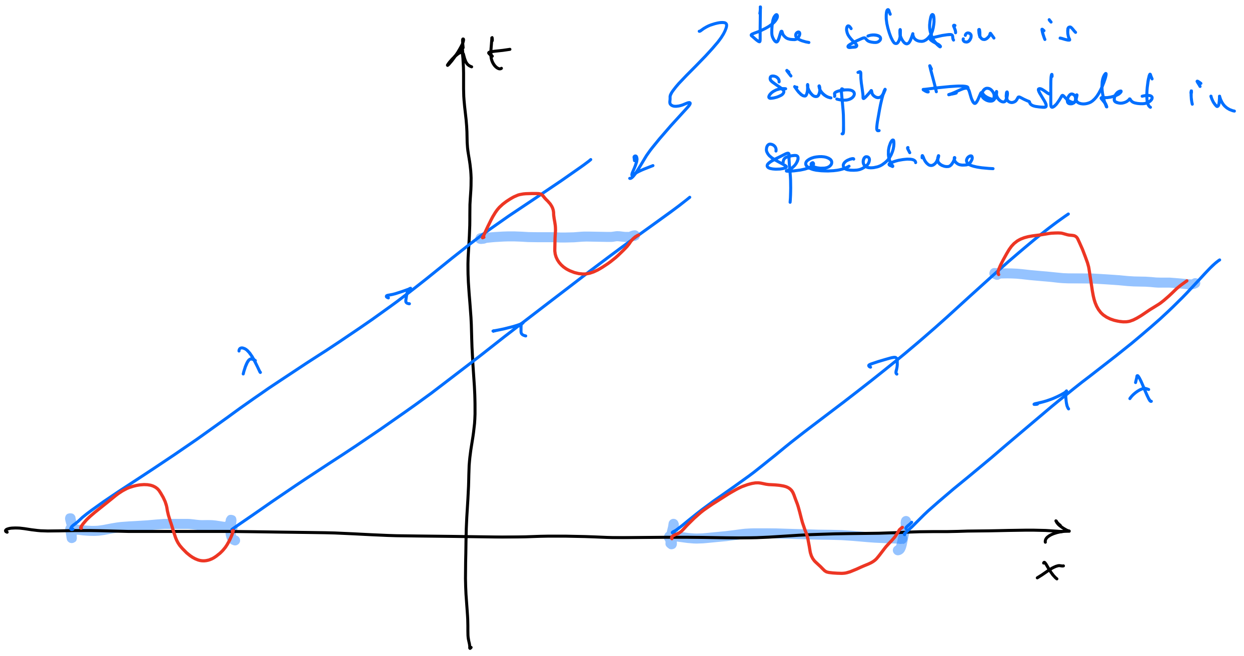
In these cases, the characteristic directions, that is, the directions in spacetime where perturbations propagate, are straight lines that do not intersect and actually allow for a "characteristic representation" of the PDE.

For example: advection equation in 1+1 D

(1) $$\boxed{\partial_t u + v\,\partial_x u = 0} \qquad v \in \mathbb{R}\,, \quad v = \text{const}$$

(1) $\Longleftrightarrow$ $$\frac{du}{dt} = 0 = \partial_t u + \lambda\,\partial_x u = 0 \qquad (2)$$

for $\boxed{\lambda = \frac{dx}{dt} = v = \text{const.}}$: characteristic direction along which u is conserved



This picture changes significantly in the case of a nonlinear problem, such as the inviscid Burger equation:

$$(3) \qquad \boxed{\partial_t u + u \partial_x u = 0}$$

In this case:

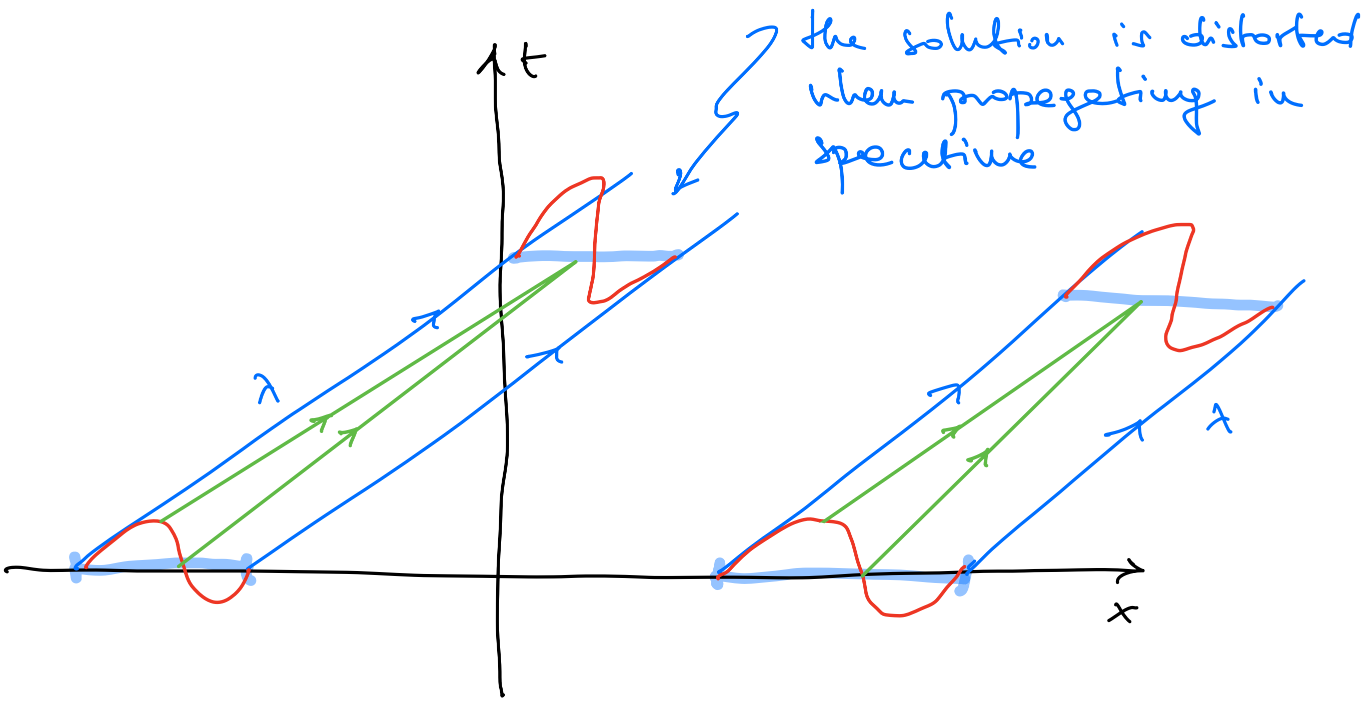
$$\boxed{\frac{du}{dt} = 0 = \partial_t u + \lambda \partial_x u = 0}$$

but $\lambda = \frac{dx}{dt} = u(x,t) \neq \text{const}$: the characteristic direction is now a function of position in spacetime.

the characteristic direction is now a function of position in spacetime.

The fact that characteristics change leads to a distorsion of the data u as it moves in spacetime; when characteristics meet, a "caustic" develops in spacetime, ie a discontinuity.

As a result, even smooth initial data may lead to a discontinuity.



Stated differently, "shocks" (ie mathematical discontinuities) are natural consequences of nonlinear hyperbolic PDEs.

This statement has three important consequences.

1. We need numerical methods that can handle discontinuous data.

2. Not all methods seen so far are able to handle discontinuous data.

3. There are two important theorems that provide a clear picture of what to do and what not to do [see lecture 9].

Theorem 1: "Non conservative schemes do not converge to the correct solution if a discontinuity is present in the data". (Hou & Le Floc, 1992)

Theorem 2: "Conservative numerical methods, if convergent, do converge to the weak (integral) solution of the problem even if the data is discontinuous". (Lax & Wendroff, 1960).

CB

Both of these theorems stress the importance of a conservative formulation of the hyperbolic PDE.

I recall that a hyperbolic equation for a variable (vector of variables) u ($\underline{u}$) is in conservative form if written as:

$$\boxed{\partial_t \underline{u} + \nabla \underline{F}(\underline{u}) = \underline{S}} \qquad (4)$$

$\underline{u}$: state vector; $\underline{F}$: flux vector, with $\underline{F} = \underline{F}(\underline{u})$

$\underline{S}$: source vector, with $\underline{S} = \underline{S}(\underline{u})$ but not of its derivatives.

1+1 problem

If $\underline{u} = u$ and we are considering only one spatial dimension, then (4) $\Longleftrightarrow$

$$\boxed{\partial_t u + \partial_x \underline{F}(u) = \cancel{\underline{S}} = 0} \qquad (5)$$

where we have set to zero the source function (it doesn't play a role in determining the mathematical character of the equation).

Let's consider the conservative formulation for the simplest nonlinear hyperbolic equation we have encountered, ie

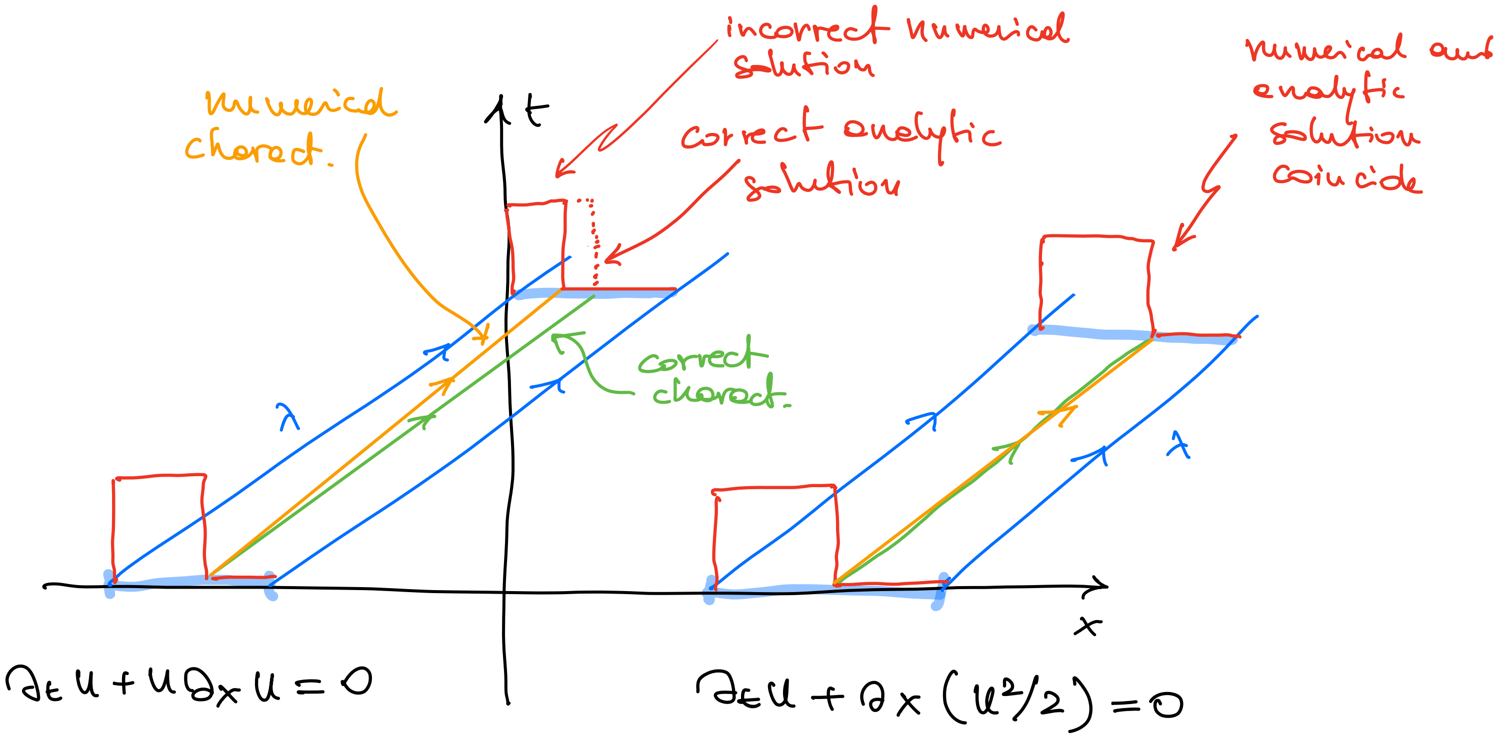
$$\boxed{\partial_t u + u\,\partial_x u = 0} \qquad (3)$$

In this case, the conservative formulation of (3) is very easy to obtain and is given by:

$$\boxed{\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0} \qquad (6)$$

where $\underline{F}(\underline{u}) = F(u) = \frac{1}{2}u^2$. Clearly, (3) and (6) are mathematically identical. Yet, their discretised solutions can be radically different!

It is instructive to appreciate the implications of the two theorems above to solve both (3) and (6) with a simple first-order method, eg upwind, when starting already from discontinuous initial data, eg a step function.



It's therefore essential to write nonlinear hyperbolic equations in a conservative form!

We have also commented how the hydrodynamical equations are perfect examples of nonlinear hyperbolic equations leading to discontinuities, ie, "shocks"!

Similarly, the equations of GRHD or GRMHD are normally written in the so-called Valencia conservative formulation, in recognition to the ideas suggested by the Valencia group led by J.-M. Ibanez in the 90's.

In such a 3+1 formulation, they are given by

$$\boxed{\partial_t \left(\sqrt{\gamma}\, \underline{U}\right) + \partial_i \left(\sqrt{\gamma}\, \underline{F}^i(\underline{U})\right) = \underline{S}(\underline{U})}$$

where

$$\underline{U} := \begin{pmatrix} D \\ S_j \\ E \end{pmatrix} = \begin{pmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - p \end{pmatrix}$$

$$\underline{F}^i := \begin{pmatrix} \alpha v^i D - \beta^i D \\ \alpha S^i_j - \beta^i S_j \\ \alpha S^i - \beta^i E \end{pmatrix}$$

$$\underline{S} := \begin{pmatrix} 0 \\ \frac{1}{2}\alpha S^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - E\,\partial_j \alpha \\ \alpha S^{ij} K_{ij} - S^j \partial_j \alpha \end{pmatrix}$$

where

$$ds^2 = -(\alpha^2 + \beta^i \beta_i)\,dt^2 + 2\beta_i\, dx^i dt + \gamma_{ij}\, dx^i dx^j$$

$$S_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta} \; ; \; S_\mu := -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta}$$

$$K_{\mu\nu} := -\gamma^\alpha{}_\mu \nabla_\alpha n_\nu$$

$$n_\mu := -\alpha \nabla_\mu t \qquad ; \quad W := (1 - v^i v_j)^{-1/2}$$

The variables in the state vector are called for obvious reasons, the "conserved" variables and are related to the (physical) "primitive" variables via their non-invertible definition:

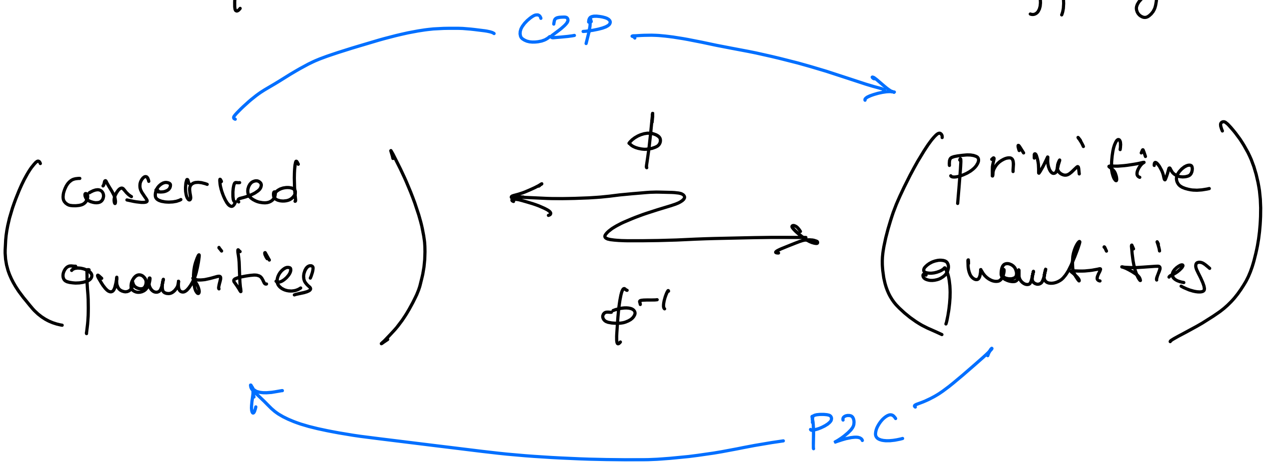
$$\begin{cases} D := \rho\, \alpha u^t = \rho W \\ S_j := \rho h u^t u_j = \rho h W^2 v_j / \alpha \\ E := \rho u^t \epsilon = \alpha \rho W \epsilon \end{cases}$$

$$\alpha u^t = W$$

$$u_j = \alpha u^t v_j = W v_j$$

Note that the new evolution system (7)-(8) does not evolve the "primitive" variables $\{\rho, u, \epsilon\}$ but the "conserved" variables $\{D, S_j, E\}$. This is a common feature: the system of equation actually solved for is not of the same physical quantities for which the physics/microphysics is based.
Rather, they are simply new variables with no specif physical meaning (at least in general) and that serve the scope of yielding a system of equations in conservative form.

This requires a continuous mapping



The mappings Φ and Φ^{-1} are not necessarily analytic and this requires the solution of a nonlinear algebraic equation (root finding).

This indeed the case in relativistic hydrodynamics and magnetohydrodynamics, where entire papers are written just about these mappings.

Numerical Methods for linear PDEs

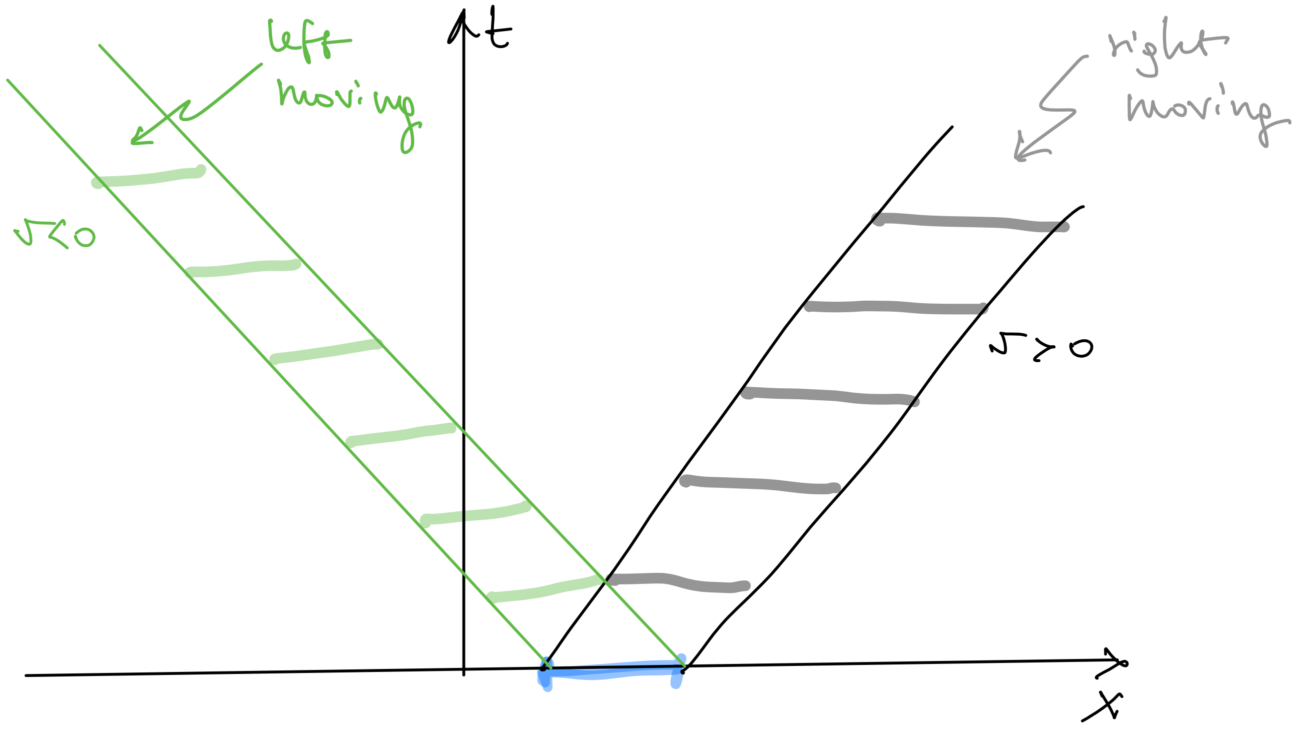
Hereafter, we will consider the simplest hyperbolic PDE in 1+1 dimensions: linear advection equation

$$\boxed{\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0 \qquad v = \text{const.}} \tag{4}$$

Because (4) is linear, it has a simple analytic solution

$$\boxed{u(x,t) = f(x - vt).}$$

In other words, the solution is simply advected in space and time



Let's discretise the differential operators ∂_t, ∂_x starting with a scheme that $O(\Delta t, \Delta x)$

Let's Taylor expand around the spacetime point (x_j, t^n)

$$u(x_j, t^n + \Delta t) = u(x_j, t^n) + \frac{\partial u}{\partial t}(x_j, t^n)\,\Delta t + O(\Delta t^2)$$

$$\Longleftrightarrow$$

$$u_j^{n+1} = u_j^n + \left.\frac{\partial u}{\partial t}\right|_j^n \Delta t + O(\Delta t^2)$$

We can isolate the differential operator

$$\underbrace{\left.\frac{\partial u}{\partial t}\right|_j^n}_{\mathcal{L}(u)} = \underbrace{\frac{u_j^{n+1} - u_j^n}{\Delta t}}_{L_h(u_j^n)} + \underbrace{O(\Delta t)}_{\epsilon_T} \qquad (5)$$

1st-order operator

In general, given a differential operator

$$\mathcal{L}(u) = \frac{\partial^m u}{\partial x^m}\bigg|_{x_j^n} = \frac{\partial^m u}{\partial x^m}\bigg|_j^n = \underbrace{L_h(u_j^n)}_{\text{p-th order discretisation of } \mathcal{L}(u)} + O(\Delta x^p)$$

In exactly the same manner, but Taylor expanding in space, we can write that

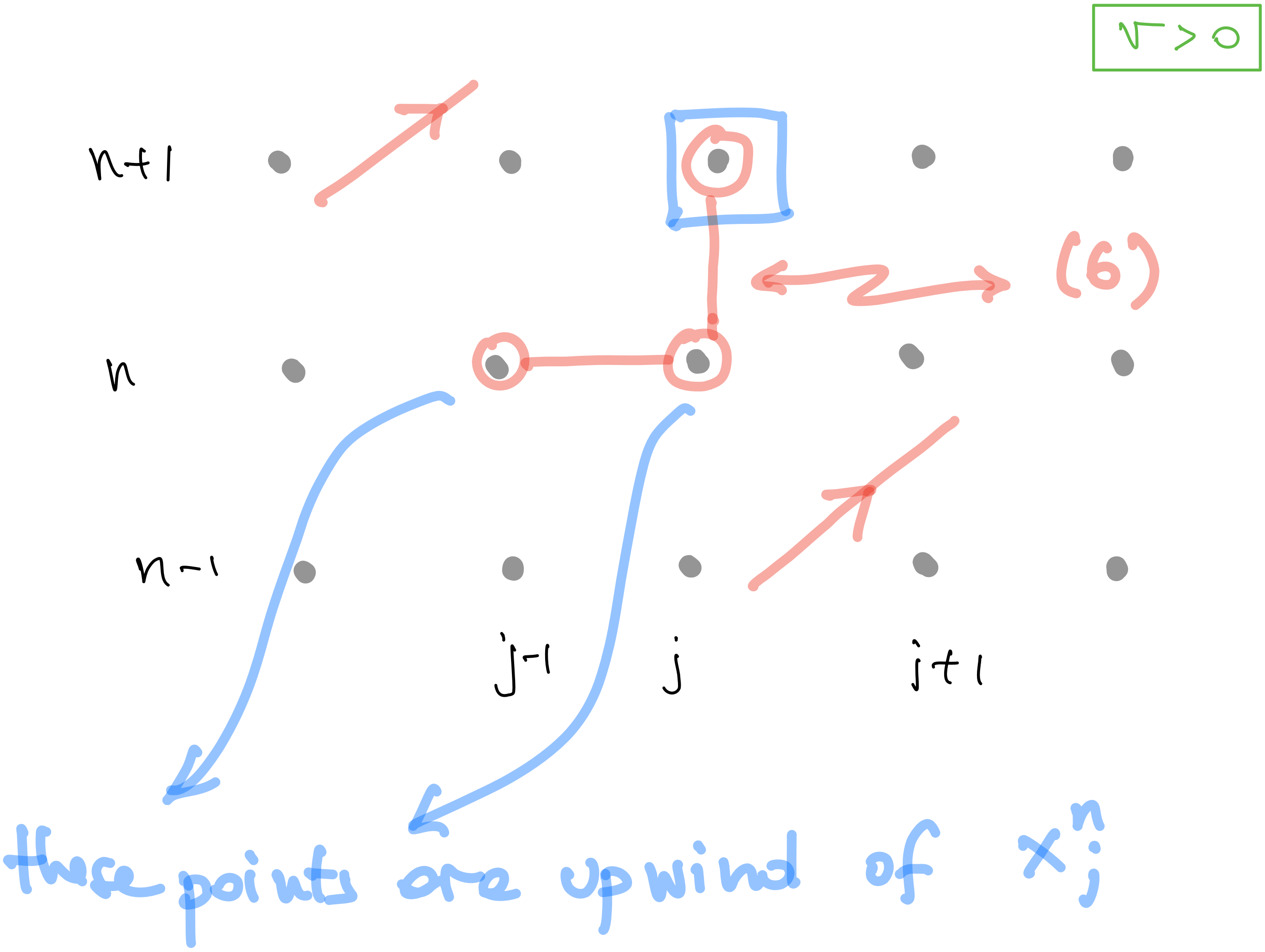
$$\frac{\partial u}{\partial x}\bigg|_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + O(\Delta x) \qquad (6)$$

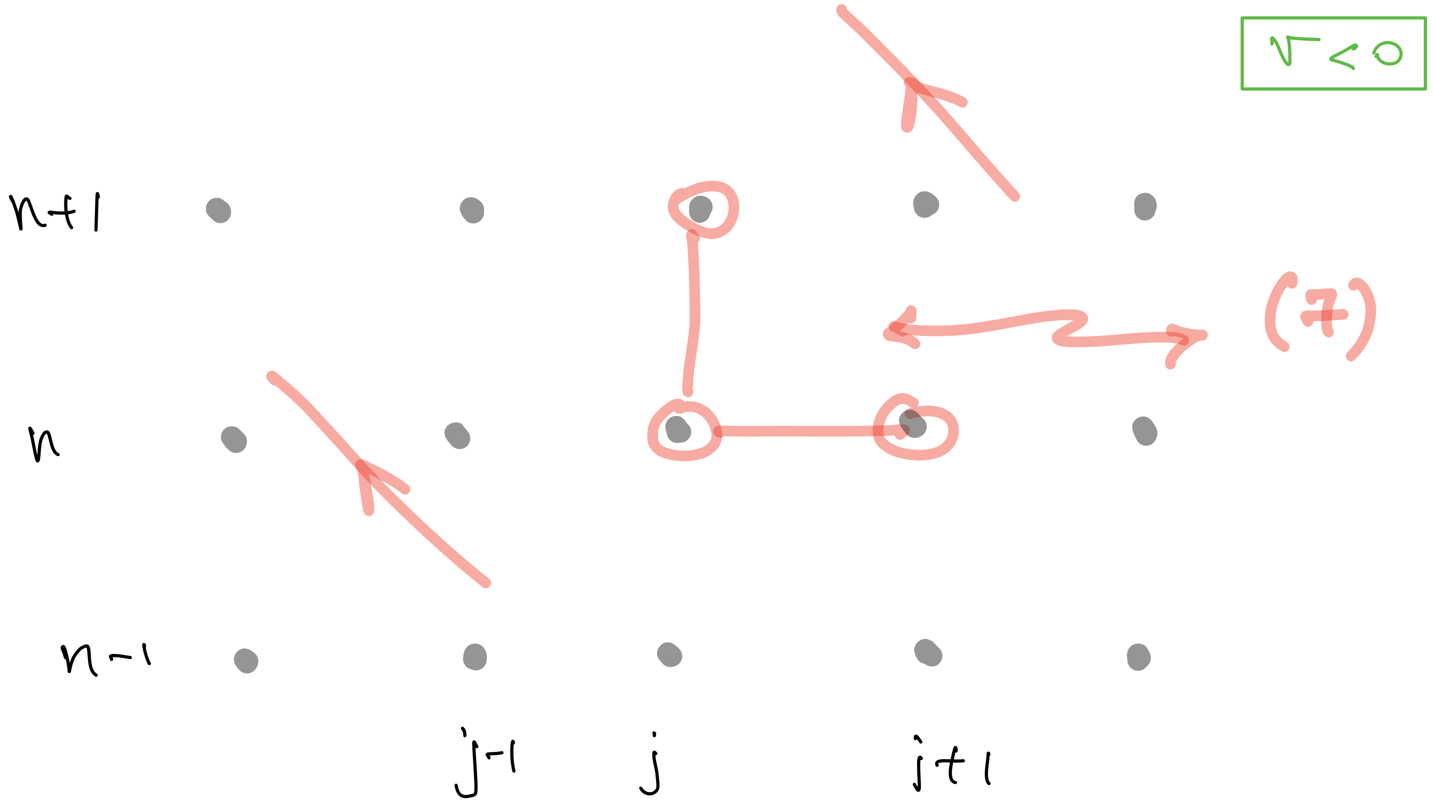
Expression (6) is not the only 1st-order representation of the spatial partial derivative on the LHS!

We could equally write

$$\left.\frac{\partial u}{\partial x}\right|_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + O(\Delta x) \qquad (7)$$

(6) and (7) are mathematically equivalent and the consequence of the fact that 1st-order "stencils" are not centered around x_j.





In other words, we can break the ambiguity by simply selecting the stencil that is upwind of the desired solution at x_j^n.

Stated differently, the advection eq. (4) admits two different discretisations depending on the sign of the advection velocity v.

$$\underbrace{\frac{u_j^{n+1}-u_j^n}{\Delta t} = -v\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) + O(\Delta t, \Delta x) \quad \text{if } v>0 \qquad \frac{u_j^{n+1}-u_j^n}{\Delta t} = -v\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) + O(\Delta t, \Delta x) \quad \text{if } v<0}_{L_h(u_j^n) \text{ of } \mathcal{L}(u)}$$

We can rewrite the two expressions above explicitly as the upwind scheme

$$\boxed{\begin{aligned} u_j^{n+1} &= u_j^n - v\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) + O(\Delta t^2, \Delta t\Delta x) \quad v>0 \\ u_j^{n+1} &= u_j^n - v\frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right) + O(\Delta t^2, \Delta t\Delta x) \quad v<0 \end{aligned}} \quad (8)$$

Normally it is not difficult to determine which of the two stencils need to be implemented even if the equations are more complex. All I is needed is to recognize what plays the role of v and compute the sign of $v(x,t)$.

Second-order in space schemes $O(\Delta t, \Delta x^2)$

Upwind is a first-order scheme. We can obtain a higher order scheme by simply improving the truncation error of the discretised spatial differential operator

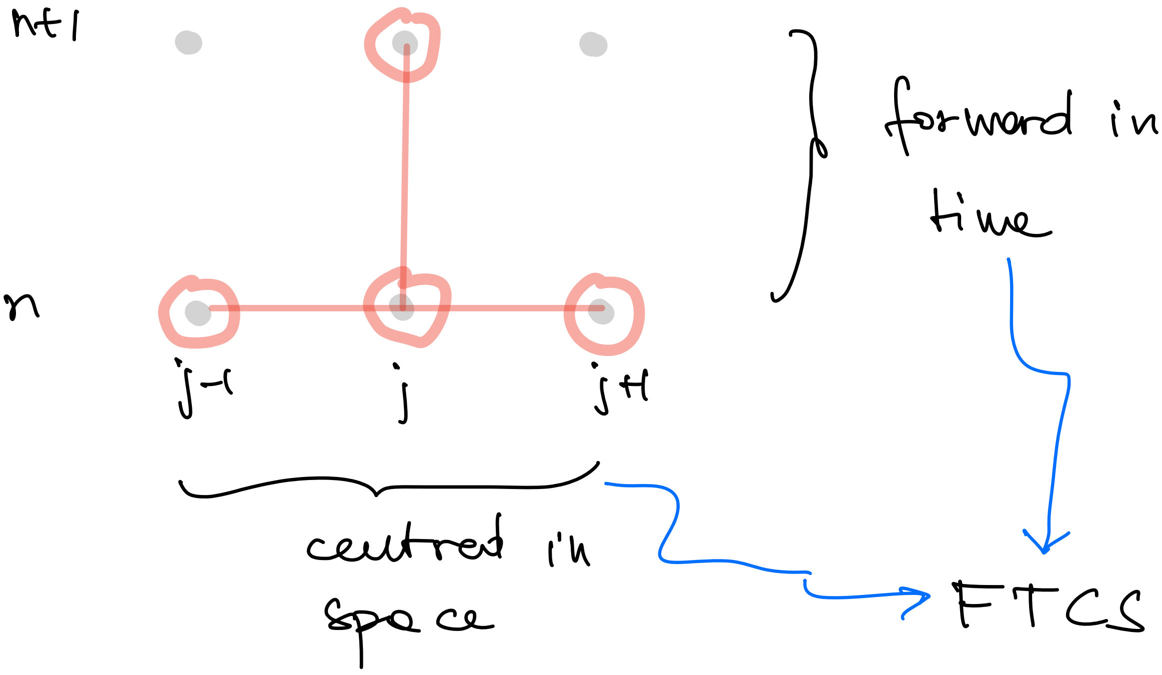
In other words, we could Taylor-expand at second order

$$(*)\quad \begin{cases} \underbrace{u(x_j+\Delta x, t^n)}_{u^n_{j+1}} = u(x_j, t^n) + \frac{\partial u}{\partial x}(x_j, t^n)\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 \\ \qquad + O(\Delta x^3) \\ \underbrace{u(x_j-\Delta x, t^n)}_{u^n_{j-1}} = u(x_j, t^n) - \frac{\partial u}{\partial x}(x_j, t^n)\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 \\ \qquad + O(\Delta x^3) \end{cases}$$

We can subtract these two expressions to obtain

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2)}$$

2nd order discretised operator.



In this way, eq (4) becomes

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2, \Delta t)$$

from which we obtain

$$\boxed{u_j^{n+1} = u_j^n - \frac{\alpha}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + O(\Delta t^2, \Delta t\,\Delta x^2)} \tag{12}$$

□

Before taking this as a useful method, I need to check that is stable

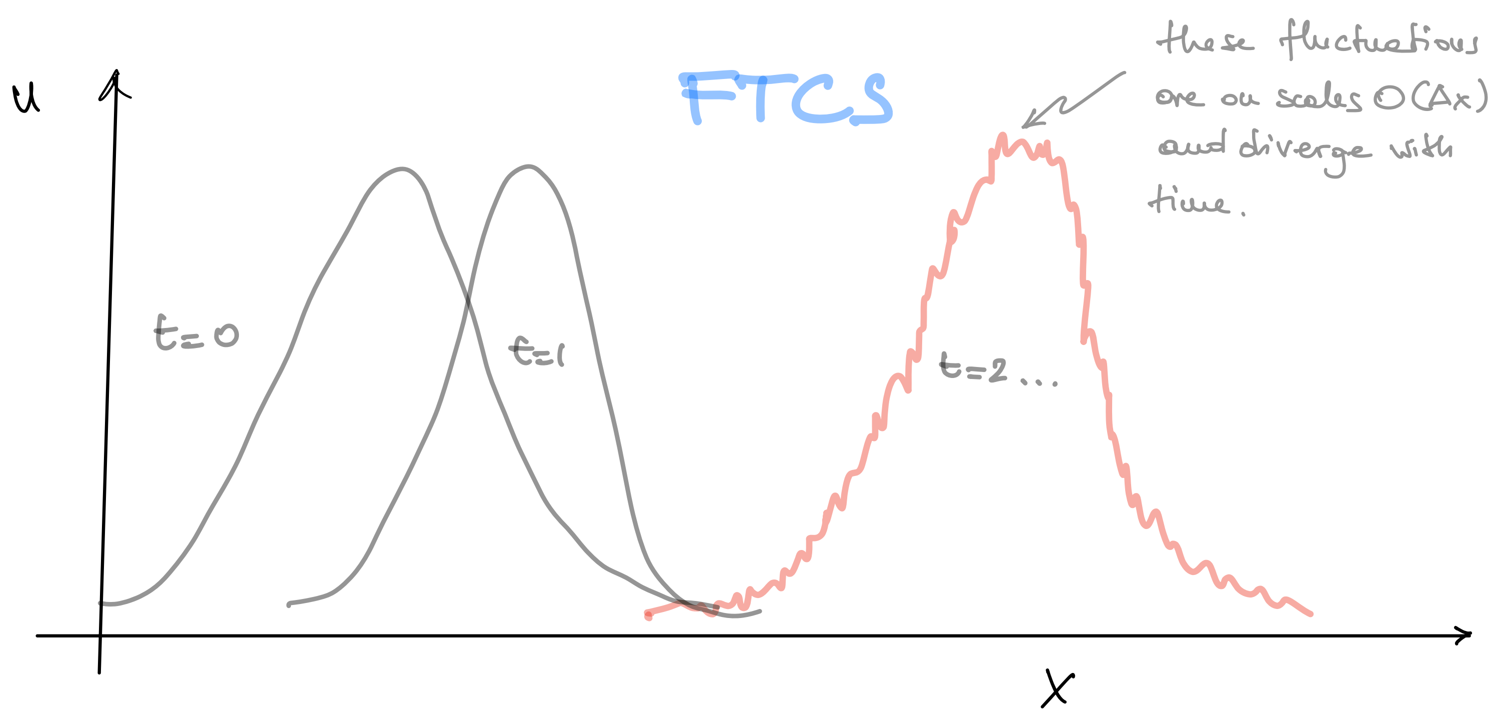
Applying the ansatz (9) to (12) we obtain

$$\xi = 1 - i\alpha \sin(k\Delta x)$$

so that $|\xi|^2 = 1 + (\alpha \sin(k\Delta x))^2 > 1$

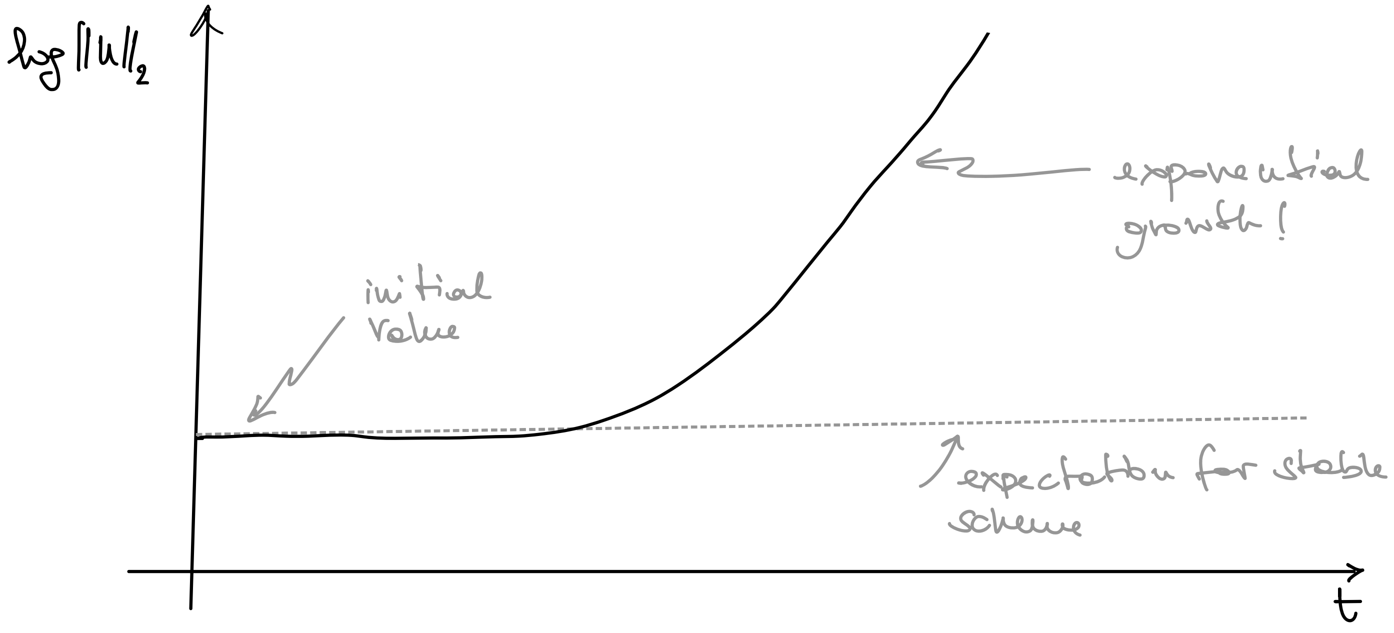
In other words, FTCS is an unconditionally unstable representation of the differential operator!

Imagine that we are actually solving the advection equation using the FTCS method. Then, our solution would look like:



these fluctuations are on scales $O(\Delta x)$ and diverge with time.

The norm, eg L2 norm, would look like this:



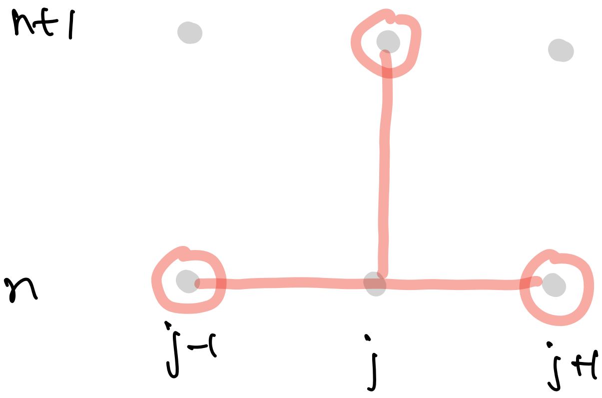
exponential growth!

initial value

expectation for stable scheme

Luckily, there is a way to fix this...

Let's see if we can fix FTCS and thus have a stable second-order scheme



Ideally we want to remove the solution at the point x^n_j, i.e., u^n_j

$$u^n_j := \frac{1}{2}\left(u^n_{j+1} + u^n_{j-1}\right) \quad : \text{spatial average}$$

and I can then rewrite FTCS as

$$\boxed{u^{n+1}_j = \overbrace{\frac{1}{2}\left(u^n_{j+1} + u^n_{j-1}\right)}^{\text{average}} - \frac{\alpha}{2}\left(u^n_{j+1} - u^n_{j-1}\right) + O(\Delta t^2, \Delta t \Delta x^2)} \qquad (13)$$

This is the Lax–Friedrichs (LF) scheme.

The amplification factor is:

$$|\xi|^2 = 1 - \sin^2(k\Delta x)(1-\alpha^2) \leq 1$$

In other words, the LF scheme is conditionally stable for $|\alpha| \leq 1$, i.e., it is CFL stable.

In order to understand why this is possible we need to realize that (13) is not the discretised representation of the advection equation (4) but of a dissipative advection equation. $\square$

Proof

Let us write (13) in a slightly different form:

$$\frac{u_j^{n+1} - u_j^n}{\cancel{\Delta t}} = -v \frac{\left(u_{j+1}^n - u_{j-1}^n\right)}{2\Delta x} + \frac{1}{2}\left(\frac{\overbrace{u_{j+1}^n - 2u_j^n + u_{j-1}^n}^{\text{average}}}{\cancel{\Delta t}}\right) \qquad (13')$$

The claim is now that eq. (13) and thus (13') is actually the representation of the advection-diffusion equation

$$\boxed{\partial_t u + v\,\partial_x u = \mathcal{L}(u)} \qquad (\cdot)$$

where $\mathcal{L}(u) := \frac{1}{2}\left(\frac{\Delta x^2}{\Delta t}\right)\partial_x^2 u$

To prove this, we can go back to expressions (*) and add rather than subtract them.

$$\begin{cases} u^n_{j+1} = u^n_j + \dfrac{\partial u}{\partial x}(x_j, t^n)\Delta x + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}\Delta x^2 + O(\Delta x^3) \\ \\ u^n_{j-1} = u^n_j - \dfrac{\partial u}{\partial x}(x_j, t^n)\Delta x + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}\Delta x^2 + O(\Delta x^3) \end{cases} \qquad (*)$$

In this way we would obtain that

$$u^n_{j+1} + u^n_{j-1} = 2u^n_j + \partial^2_x u\Big|^n_j \Delta x^2 + O(\Delta x^3)$$

so that

$$\boxed{\left.\frac{\partial^2 u}{\partial x^2}\right|^n_j = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\Delta x^2} + O(\Delta x^2)} \qquad (\cdot\cdot)$$

Expression $(\cdot\cdot)$ is the second-order discretisation of the second-order partial derivative.

This proves the statement that (13) is the finite-difference representation of the advection-diffusion equation

$$(14) \quad \boxed{\partial_t u + v\,\partial_x u = \underbrace{D\,\partial_x^2 u}}$$

part responsible for the dissipative nature of the advection equation.

$$D := \frac{1}{2}\,\frac{\Delta x^2}{\Delta t}$$

Q. Is this really allowed? Why?

We have seen that the advection equation is rewritten as the equation $(\cdot)$

$$\underbrace{\partial_t u + v\,\partial_x u}_{O(\Delta x)} = \underbrace{\frac{1}{2}\left(\frac{\Delta x^2}{\Delta t}\right)\partial_x^2 u}_{O(\Delta x^3)} \qquad (\cdot)$$

The coefficient in front of the second derivate is simply

$$\frac{\Delta x^2}{\Delta t} = O(\Delta x)$$

since $\Delta t = O(\Delta x)$ by the CFL condition and the second partial derivative is $O(\Delta x^2)$.

As a result, the whole RHS of (·) is

$$\frac{1}{2}\left(\frac{\Delta x^2}{\Delta t}\right)\partial_x^2 u = O(\Delta x \cdot \Delta x^2) = O(\Delta x^3)$$

The LHS, instead is $O(\Delta x)$ because

$$\partial_t u = \underset{\substack{\uparrow \\ \text{discretised version} \\ \text{of operator}}}{\partial_t u^{(h)}} + O(\Delta x)$$

$$\partial_x u = \partial_x u^{(h)} + O(\Delta x^2)$$

so that, when concentrating on the order of the truncation error we have

$$\partial_t u^{(h)} + O(\underbrace{\Delta x) + v\,\partial_x u^{(h)} + O(\Delta x^2}_{O(\Delta x)}) = O(\Delta x^3)$$

This means that the RHS of (.) goes to zero more rapidly than the error in the LHS

In the limit of $\Delta x \to 0$, equation (.) does converge to the continuum advection equation

In other words, mathematically speaking, (.) is an acceptable and convergent approximation to the original advection equation (4).

There is also another way of reading on the validity of the LF scheme and it is contained in the amplification factor, which we recall is

$$|\xi|^2 = 1 - \sin^2(k\Delta x)(1-\alpha^2)$$

For $\alpha \neq 1 \iff |v|\Delta t \neq \Delta x$, then $|\xi|^2 < 1$, in other words, the solution will be dissipated (leak of energy).

Note that this dissipation is proportional to $\sin^2(k\Delta x)$ and hence it will effect differently the various wavelengths involved in the solution.

In particular, the dissipation will be very small if

$$k\Delta x \ll 1$$

where k is

$$k = \frac{2\pi}{\lambda} \simeq \frac{2\pi}{u/u'}$$

$\lambda := \frac{u}{u'}$: typical length scale (scale-height) of the problem at hand

As a result, the product $k\Delta x = 2\pi \frac{\Delta x}{\lambda} \ll 1$ if

$$\boxed{\lambda \gg \Delta x}$$

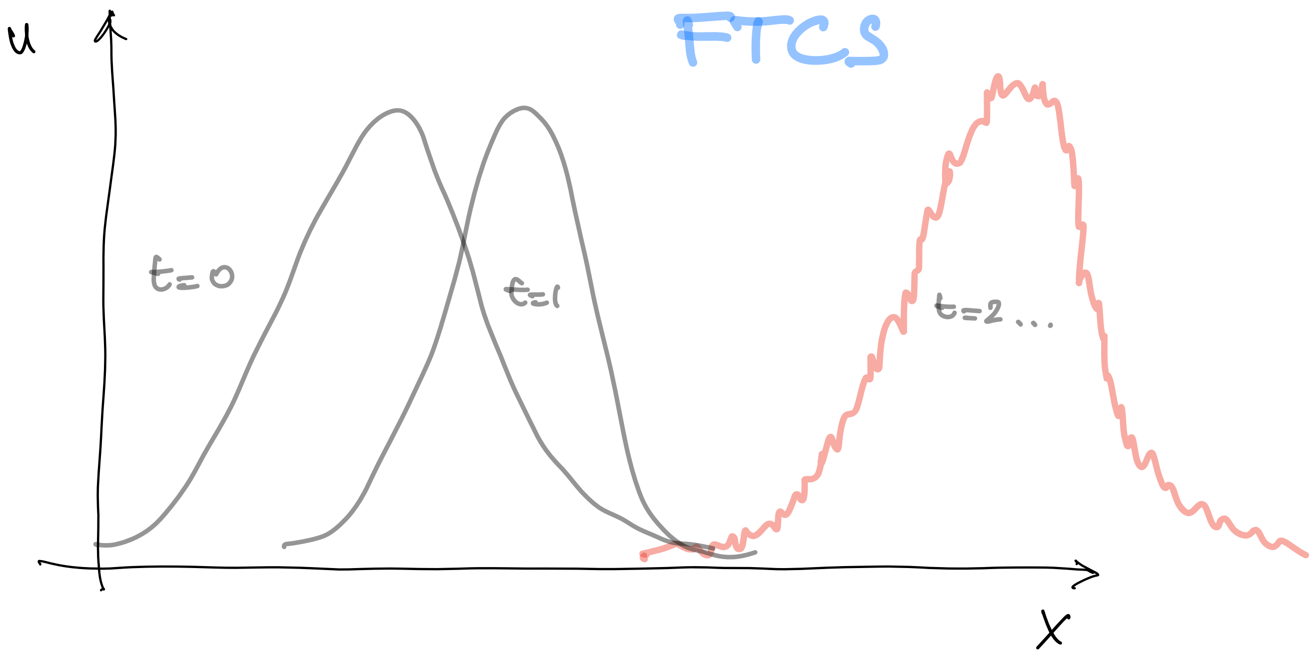
In other words, the dissipation is small on scales λ that are much larger than Δx, ie large-scale features.

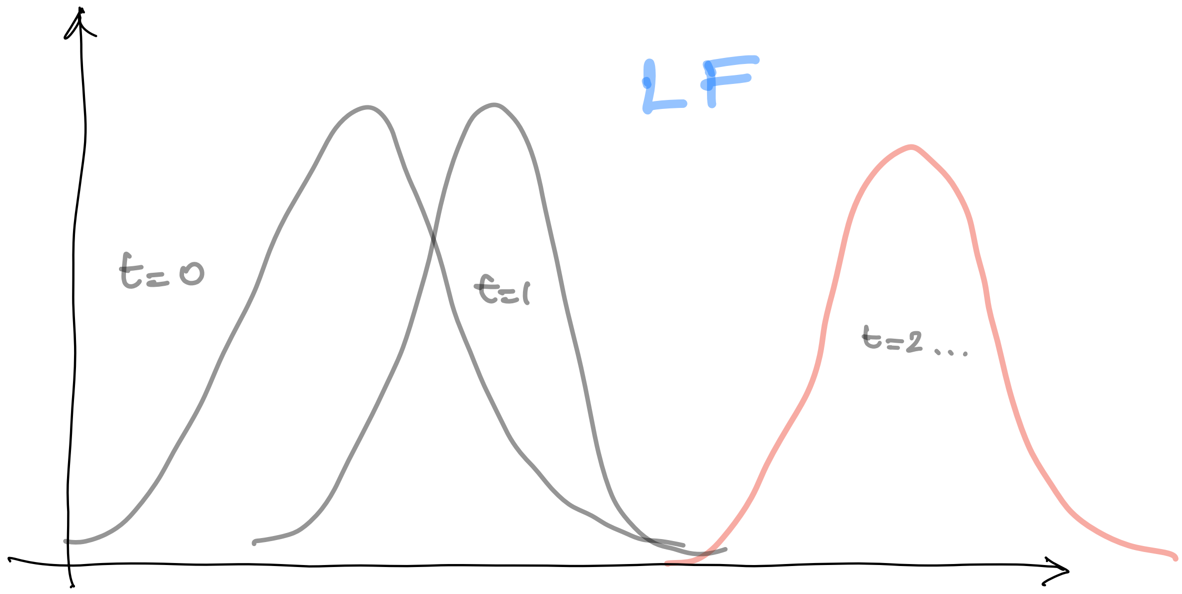
By contrast, the dissipation will be large for small-scale features, ie for

$$k' \sim \Delta x \iff \lambda' \sim \Delta x.$$

This is an effective scale dependent dissipation.

These "fine" features of the solutions will be dissipated and this is where the FTCS instability would manifest.





EXTRA

On the classification of (2nd-order) Partial Differential Equations (PDEs)

Solution of partial differential equations (PDEs)

Let's consider a generic PDE of second order in two dimensions (x,y); it can be written generically as:

$$\boxed{a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + f(x,y,u,\partial u) = 0} \quad (1)$$

a_{ij} : coefficients of the PDE

We will hereafter consider PDEs of second-order because in general this is the highest order at which PDEs appear in physics. Even the most complex ones, eg, the Einstein equations, are second-order PDEs.

If the order is higher, it is still possible to recast the problem into a set of PDEs of lowest order, eg, first-order.

In this case, eq. (1) is said to be a linear PDE with variable coefficients.

- (1) is said to be linear with constant coefficients if

$$\boxed{a_{ij} = \text{const.}}$$

- (1) is said to be "quasi-linear" if

$$\boxed{a_{ij} = a_{ij}(x, y, u)}$$

- (1) is said to be "non linear" if

$$\boxed{a_{ij} = a_{ij}(x, y, u, \partial u, \partial^2 u)}$$

Note that this nomenclature is followed mostly in a mathematical context. Often in physics, also "quasi-linear" equations are referred to as "non linear".

CAH

The character of PDEs (1) can be classified by looking at its coefficients and in particular at the determinant of the matrix of coefficients.

$$\Delta := a_{11}a_{22} - (a_{12})^2 \overset{\text{roots of determinant}}{\downarrow} = 0 : \begin{cases} \text{Hyperbolic: roots are real and distinct} \\ \text{Parabolic: roots are real but zero} \\ \text{Elliptic: roots are complex} \end{cases}$$

Elliptic equations describe Boundary Value Problems (BVP) since the space of solutions Ω depends on the values of solution at its boundaries $\partial\Omega$ (boundary conditions).

In physics, if one of the coordinates is time, then elliptic equations are easily recognisable by the fact that they do not depend on time. The prototypical elliptic equation is the Poisson equation :

$$\boxed{\underbrace{\nabla^2}_{\partial^2_x + \partial^2_y} \phi(x,y) = \rho(x,y)}$$

determine gravitational potential ϕ once distribution of density ρ is given

Hyperbolic and parabolic equations define instead Initial-Value-Boundary problems (IVBP, or Cauchy problem) because the solution $u(x,t)$ depends on the initial value of the solution and, in principle, on the boundary values.

In physics, i.e. in PDEs describing physical laws, it is easy to recognize an IVBP because it will show a dependence on time.

A prototypical hyperbolic equation is the wave equation

$$\boxed{\Box\,\phi(x,t) = 0}$$

given $\phi(x,0)$, determines how ϕ "propagates" with speed v

where (in 1+1 dimensions)

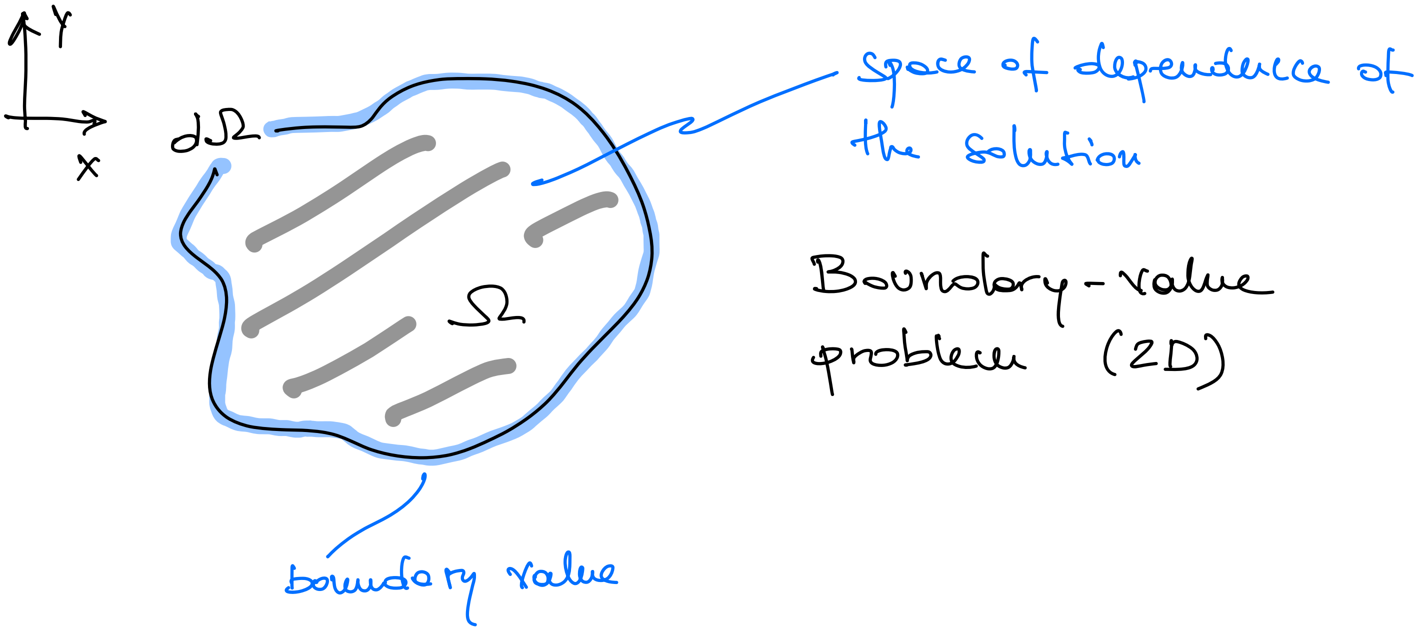
$$\Box := \partial_t^2 - v^2 \partial_x^2 = \frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2}$$

Similarly, a prototypical parabolic equation is given by the diffusion equation (also known as "heat equation")

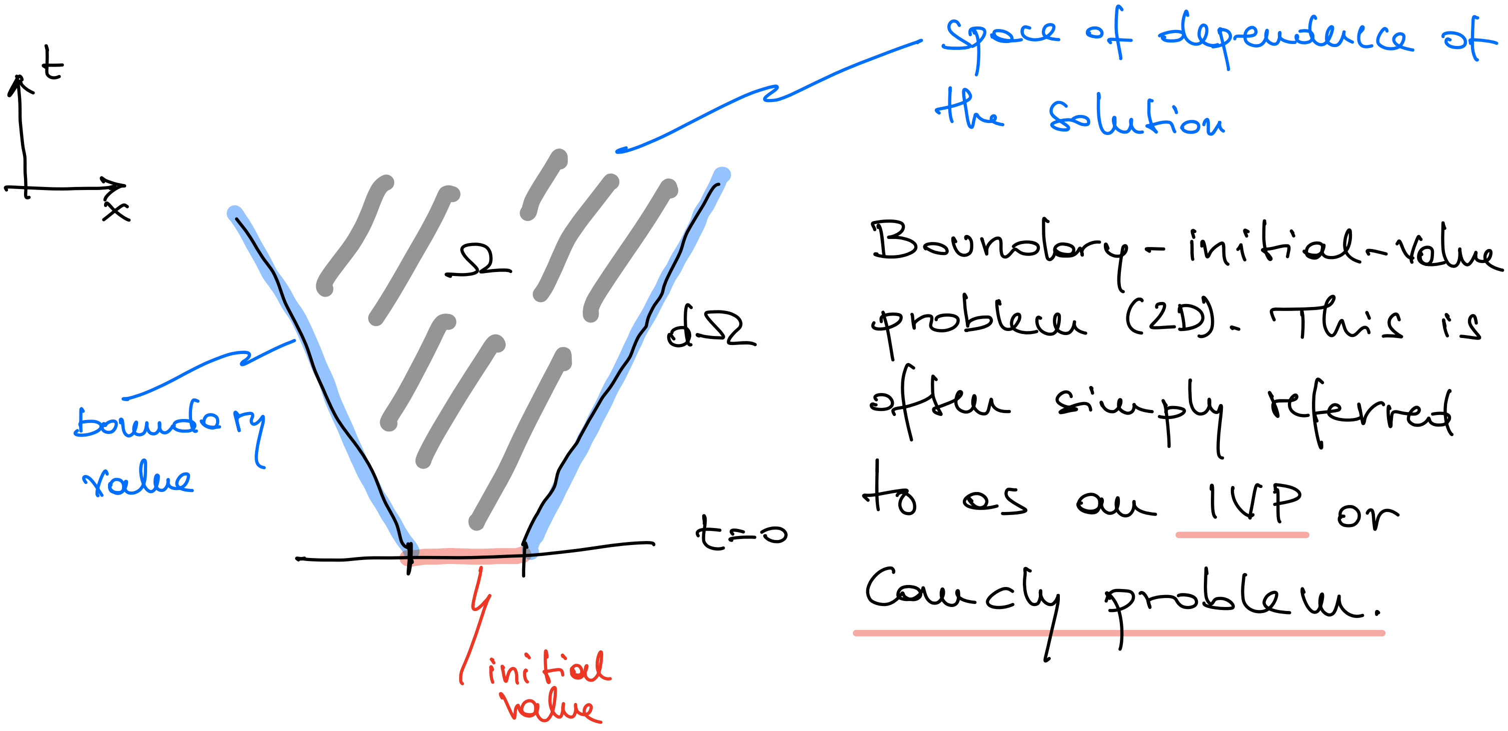
$$\boxed{\begin{aligned} \partial_t u(x,t) &= \partial_x (D\, \partial_x u) \\ &\overset{D=\text{const.}}{=} D\, \partial_x^2 u \end{aligned}}$$

given $u(x,0)$ determines how u "diffuses" with time.

We can also represent the differences between BVPs and IVBPs graphically in terms of the space of dependence of the solutions.



Boundary-value problem (2D)



Boundary-initial-value problem (2D). This is often simply referred to as an IVP or Cauchy problem.