BIP, Timisoara May 2026

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Lecture Ia: 3+1 split in general relativity

Lecture Ib: ADM formulation

Lecture IIa: Relativistic Hydrodynamics

Lecture IIb: Numerical Methods for Hyperbolic PDEs

Lecture IIIa: Application to BNS$_3$: bulk dynamics & GWs

Lecture IIIb: Application to BNS$_3$: EM emission and neutrinos

Numerical Methods for nonlinear PDEs

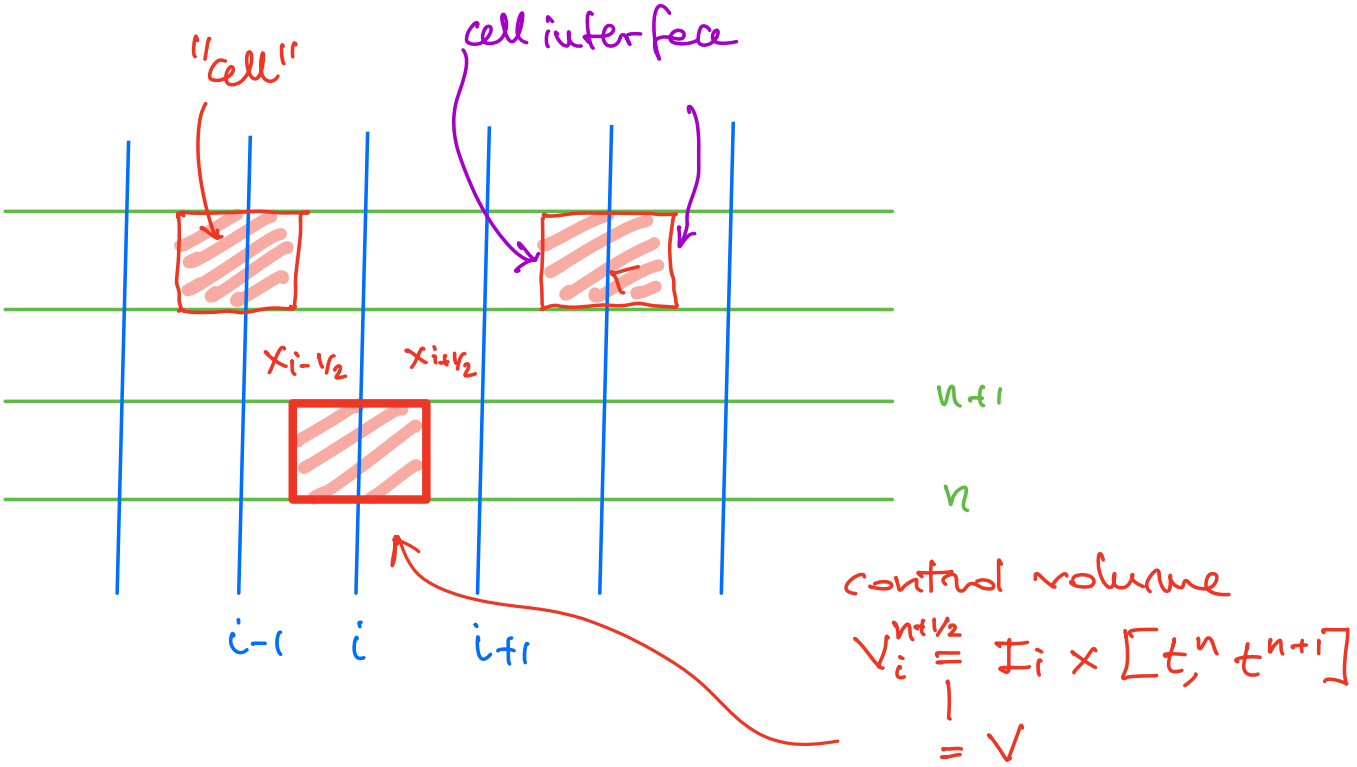
Finite-Volume Methods

Let's now proceed with the numerical solution of

$$\partial_t \underline{U} + \partial_x \underline{F}(\underline{U}) = 0 \qquad (8)$$

adopting an integral approach over a so-called "control volume".

In essence, we discretise space and time and generate small space time volumes:



Note: the cell interfaces are at $x_{i\pm 1/2}$ so that the control volume is centred around x_i. This is not necessary for the time since we do not have fluxes.

Let's first integrate (8) over space:

total derivative because of integral in space

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x,t)\,dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} \partial_x \underline{F}(u,t)\,dx =$$

$$= \underline{F}\left(\underline{U}(x_{i-1/2},t)\right) - \underline{F}\left(\underline{U}(x_{i+1/2},t)\right) \tag{11}$$

and next over time

$$\int_{t^n}^{t^{n+1}} \frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x,t)\,dx\,dt = \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x,t^{n+1})\,dx - \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x,t^{n})\,dx \tag{12}$$

$$= \int_{t^n}^{t^{n+1}} \left(\underline{F}\left(\underline{U}(x_{i-1/2},t)\right) - \underline{F}\left(\underline{U}(x_{i+1/2},t)\right)\right) dt$$

$$= \int_{t^n}^{t^{n+1}} \underline{F}\left(\underline{U}(x_{i-1/2},t)\right) dt - \int_{t^n}^{t^{n+1}} \underline{F}\left(\underline{U}(x_{i+1/2},t)\right) dt$$

Dt

We can now introduce spatial averages

$$\underline{U}_i^n := \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x, t^n)\, dx \tag{13}$$

$$\underline{F}_{i\pm 1/2} := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \underline{F}\left(\underline{U}\left(x_{i\pm 1/2}, t\right)\right) dt \tag{14}$$

Note that in general these are "volume" averages, hence the name finite-volume methods.

Using these averages, Eq. (11) can be written as

$$\underline{U}_i^{n+1} = \underline{U}_i^n + \frac{\Delta t}{\Delta x}\left(\underline{F}_{i-1/2} - \underline{F}_{i+1/2}\right) \tag{15}$$

blue because exact

red because discretised

DS

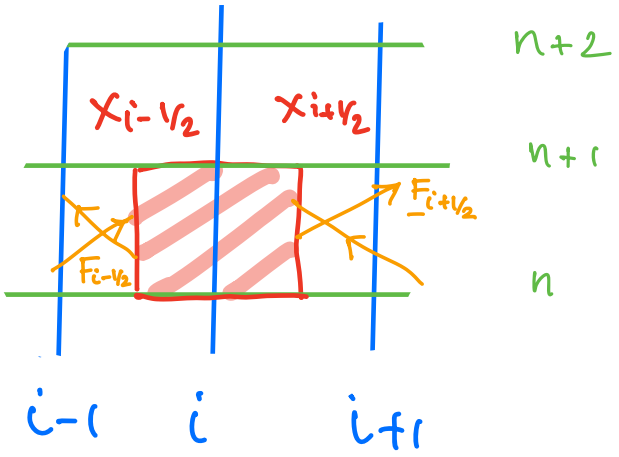
Notes :

1. Although I've used red box around (15), it is not yet a numerical scheme. Indeed (15) is an exact expression since no approximations were made! Hence it also has a blue box...

2. Equation (15) becomes a numerical method when one introduce an approximation and a numerical prescription for how to compute the fluxes $F_{i\pm 1/2}$

3. When a prescription for the evaluation of (15) is made, (15) is called a "Godunov scheme" from the Russian mathematician first suggesting it.

4. Expression (15) makes the "conservative" nature of the equation transperent: U can change in V only because of ingoing outgoing fluxes. What is lost from V goes somewhere else but it's not lost!



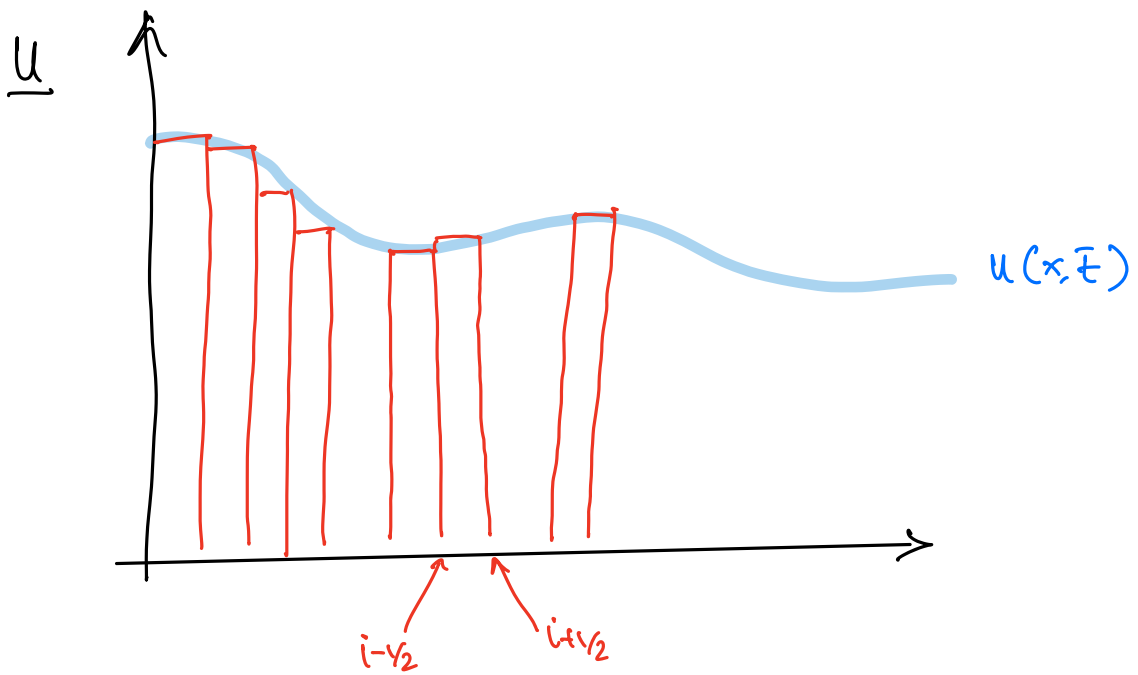
5. Different numerical algorithms can be devised to provide an ever more accurate description of the fluxes $\underline{F}_{i\pm 1/2}$ at the cell interfaces $x_{i\pm 1/2}$

6. The simplest of such algorithms is when $\underline{F}$ acts on $\underline{U}$ as an average (i.e., $F \propto \underline{U}$). In this case, the method (15) is clearly 1st-order as can be appreciated by considering the simplest case of an advection equation. In this case $\underline{F}(\underline{U}) = \lambda \underline{U}$ and Eq. (15) can be written as ($C := \lambda \Delta t / \Delta x$)

$$(16) \quad \begin{cases} \underline{U}_i^{n+1} = \underline{U}_i^n - C\left(\underline{U}_i^n - \underline{U}_{i-1}^n\right) & \text{if } \lambda > 0 \\ \underline{U}_i^{n+1} = \underline{U}_i^n - C\left(\underline{U}_{i+1}^n - \underline{U}_i^n\right) & \text{if } \lambda < 0 \end{cases}$$

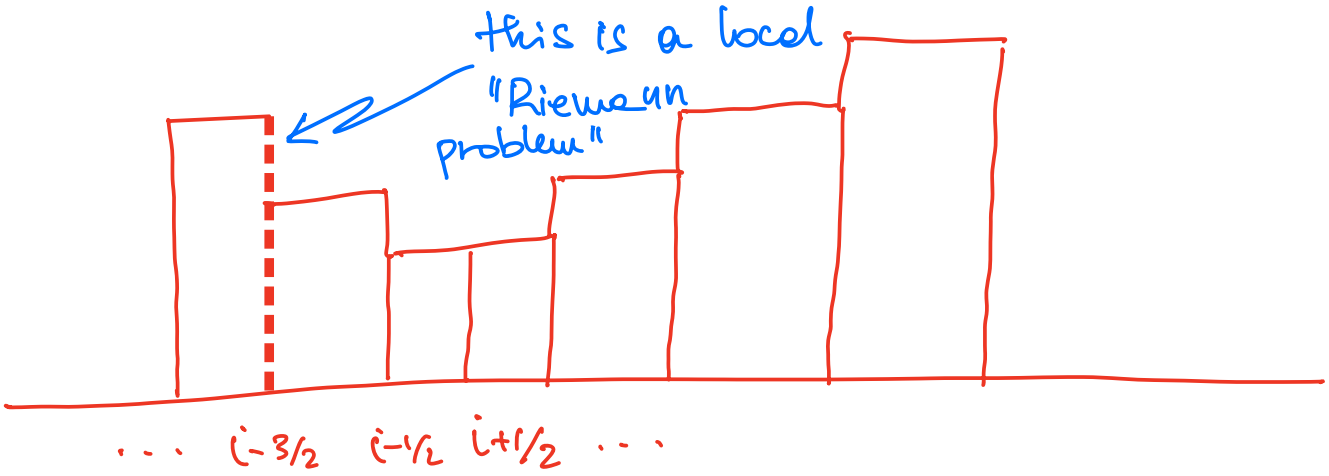
Expressions (16) are clearly the "upwind" method, that we have seen to be $O(\Delta t)$ and $O(\Delta x)$, ie a first-order method.

7. The volume averages of $\underline{U}$ in (13) imply that our quantities $\underline{U}(x,t)$ are described as a piece-wise constant sequence of constant "states"



The geniality of Godunov's approach is in recognising that this piecewise constant representation of $\underline{U}$ is equivalent to the construction of a sequence of local Riemann problems.

DB



Hence, the solution of the Riemann problem can provide an exact solution for $\underline{U}(x_{i\pm1/2}, t)$ that is, the state vector at the all interface, from which one can compute $\underline{F}(\underline{U}(x_{i\pm1/2}, t))$ exactly.

What is a Riemann problem ?

A Riemann problem is the determination of the flow pattern that develops in the presence of constant and discontinuous initial data.

This is a very old problem that Riemann formulated but did not solve also because there is no analytic solution.

From a mathematical point of view, it consists in finding the solution of the state vector of the hydrodynamic equations $\underline{U}(x,t)$ when the initial conditions are

$$\underline{U}(x,0) = \begin{cases} U_L = \text{const.} & x<0 \\ U_R = \text{const.} & x>0 \end{cases} \qquad (17)$$

As mentioned, no solution was found, but the conclusion was that the flow will in general composed of two nonlinear waves $W_{\leftarrow}$, $W_{\rightarrow}$, that are either shock waves or rarefaction waves, propagating in opposite directions and seperated by another nonlinear wave (a contact discontinuity) between two constant states.

Mathematically:

left — L R — right at $t=0$

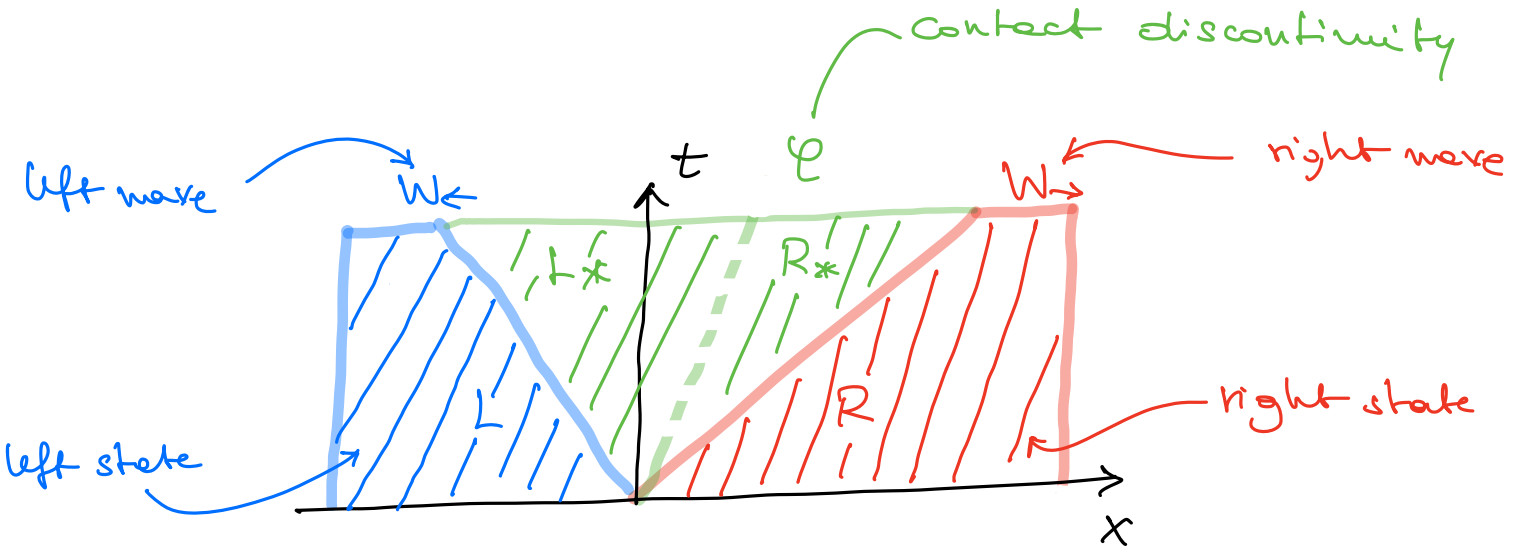
contact discontinuity

$L \; W_{\leftarrow} \; L_* \; \mathcal{C} \; R_* \; W_{\rightarrow} \; R$ at $t>0$

left wave — intermediate state (L)

(18)

Diagrammatically:



This is also called the "Riemann fan" and is composed of four possible classes of solutions:

(19)

1. $L\ \mathcal{S}_{\leftarrow}\ L*\ \mathcal{C}\ R*\ \mathcal{S}_{\rightarrow}\ R$	: two shocks
2. $L\ \mathcal{R}_{\leftarrow}\ L*\ \mathcal{C}\ R*\ \mathcal{S}_{\rightarrow}\ R$	: one shock, one rarefaction
3. $L\ \mathcal{S}_{\leftarrow}\ L*\ \mathcal{C}\ R*\ \mathcal{R}_{\rightarrow}\ R$	: one shock, one rarefaction
4. $L\ \mathcal{R}_{\leftarrow}\ L*\ \mathcal{C}\ R*\ \mathcal{R}_{\rightarrow}\ R$	: two rarefactions

Although the Riemann problem cannot be solved analytically, it can be solved "exactly" via the solution of a set of ODEs.

In turn, this provides an "exact" value of $\underline{U}_{i\pm1/2} = \underline{U}(x_{i\pm1/2}, t)$ from which the fluxes $\underline{F}(\underline{U}_{i\pm1/2}, t)$ in Eq. (15) can be computed.

So, at least in principle, the RHS of Eq. (15),

$$\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \underline{F}\left(\underline{U}(x_{i\pm1/2}, t)\right)dt =: \underline{F}_{i\pm1/2}$$

can be computed "exactly" leading to an "exact" solution of the system of conservation laws in a conservative form given by Eq. (8).

In practice, because this has to be done at each cell interface, exact Riemann solvers are never used (apart from tests) and the solution is made via approximate Riemann solvers, of which there are many different options, each with pros and cons.

HLLE Approximate Riemann Solver

All approximate Riemann solvers make some assumptions on the properties of the Riemann fan and so does HLLE.

In particular, HLLE assumes that only two waves are produced propagating in opposite directions with speeds $\lambda_L \leq 0$, $\lambda_R \geq 0$ and generating a single and constant state between them:

$$\underline{U}(x,t) = \begin{cases} \underline{U}_L & \text{if} \quad x/t < \lambda_L \\ \underline{U}^{HLLE} & \text{if} \quad \lambda_L < x/t < \lambda_R \\ \underline{U}_R & \text{if} \quad x/t > \lambda_R \end{cases} \tag{20}$$

L W_L L* $\mathcal{C}$ R_* W_R R Riemann solution

L W_L $\underline{U}^{HLLE}$ W_R R HLLE solution

the control volume is given by

$$V = [-\Gamma, \Gamma] \times [0, T]$$

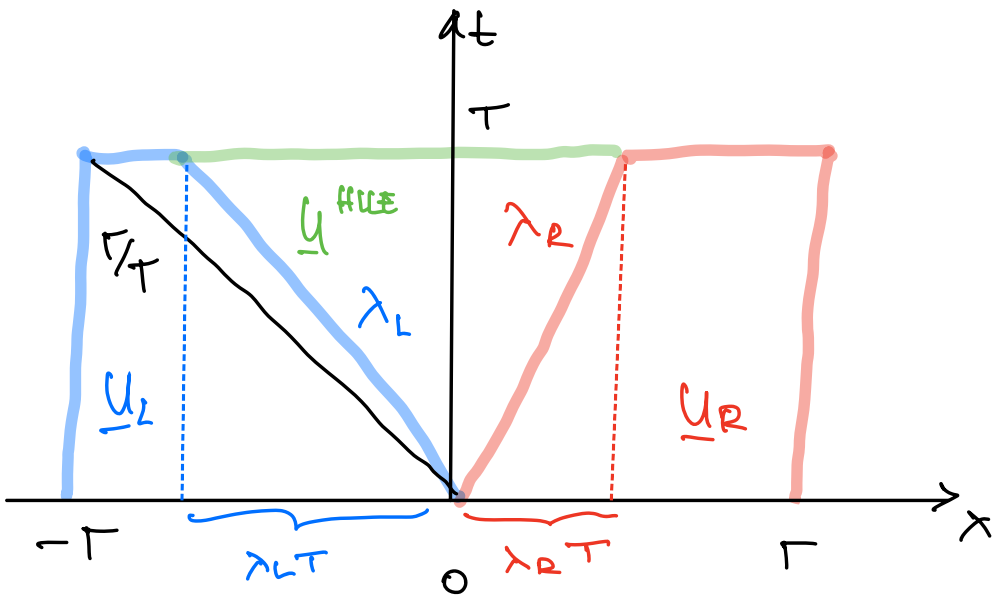
where

$$\Gamma > \max(|\lambda_L|, |\lambda_R|)\, T$$

want the wave not to move out of the cell

and T is a representative time interval, eg

$$T := t^{n+1} - t^n$$



the integral form of the conservative equation (15)

$$\underline{U}_i^{n+1} = \underline{U}_i^n + \frac{\Delta t}{\Delta x}\left(\underline{F}_{i-1/2} - \underline{F}_{i+1/2}\right)$$

where

$$\underline{U}_{i}^{n} := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x, t^n)\,dx \; ; \; \underline{F}_{i\pm 1/2} := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \underline{F}\left(\underline{U}\left(x_{i\pm 1/2}, t\right)\right) dt$$

is:

$$\int_{-T}^{T} \underline{U}(x,T)\,dx = \int_{-T}^{T} \underline{U}(x,0)\,dx + \int_{0}^{T} \underline{F}(\underline{U}(-T,t))\,dt \qquad (x_{i-1/2})$$

at the edges the state is not affected yet: $T > \lambda_{L,R}\, T$

$$- \int_{0}^{T} \underline{F}\left(\underline{U}(T,t)\right) dt \qquad (21)$$

$$= \int_{-T}^{0} \underline{U}_L\,dx + \int_{0}^{T} \underline{U}_R\,dx + \int_{0}^{T} \underline{F}(\underline{U}(-T,t))\,dt - \int_{0}^{T} \underline{F}\left(\underline{U}(T,t)\right) dt$$

$\underline{U}_L$: $+T\,\underline{U}_L$; $\underline{U}_R$: $T\,\underline{U}_R$

$$= T\left(\underline{U}_L + \underline{U}_R\right) + T\left(\underline{F}(\underline{U}_L) - \underline{F}(\underline{U}_R)\right) =$$

$$= T\left(\underline{U}_L + \underline{U}_R\right) + T\left(\underline{F}_L - \underline{F}_R\right) \qquad (22)$$

The LHS of (21) can also be split

$$\int_{-r}^{r} \underline{u}(x,T)\,dx = \int_{-r}^{T\lambda_L} \underline{u}_L\,dx + \int_{T\lambda_L}^{T\lambda_R} \underbrace{\underline{u}(x,T)\,dx}_{\underline{u}^{HLLE}} + \int_{T\lambda_R}^{r} \underline{u}_R\,dx$$

$$= \underline{u}_L\left(T\lambda_L + r\right) + \underline{u}^{HLLE} T\left(\lambda_R - \lambda_L\right) + \underline{u}_R\left(r - T\lambda_R\right)$$

(22)

$$= r\left(\underline{u}_L + \underline{u}_R\right) + T\left(\underline{F}_L - \underline{F}_R\right)$$

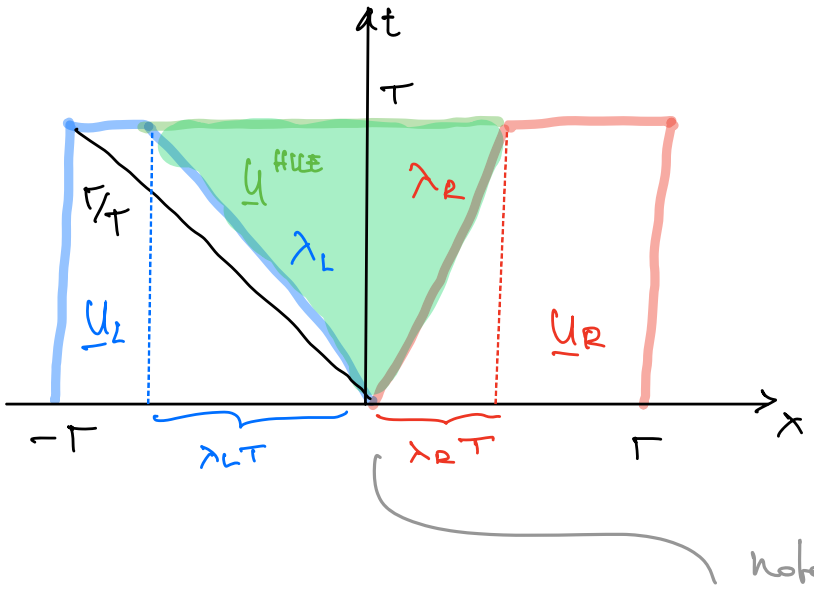
Collecting terms:

$$\underline{u}^{HLLE} = \Big[r\left(\underline{u}_L + \underline{u}_R\right) + T\left(\underline{F}_L - \underline{F}_R\right) - \underline{u}_L\left(T\lambda_L + r\right) - \underline{u}_R\left(r - T\lambda_R\right)\Big] \Big/ \left[T\left(\lambda_R - \lambda_L\right)\right]$$

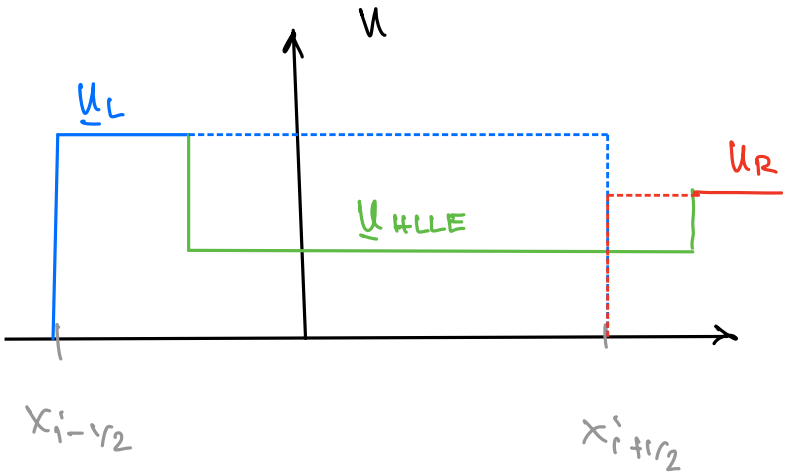
$$= \boxed{\frac{\lambda_R \underline{u}_R - \lambda_L \underline{u}_L + \underline{F}_L - \underline{F}_R}{\lambda_R - \lambda_L} = \underline{u}^{HLLE}} \qquad (23)$$

In other words, the intermediate state $\underline{U}^{HLLE}$ can be easily computed in terms of the signal speeds λ_L and λ_R and the states $\underline{U}_L$, $\underline{U}_R$ (from which one also computes $\underline{F}_L$, $\underline{F}_R$).

Referring to the respective diagram, we have just computed the "green" state at the center of the cell.



note that this point is at $x_{i+1/2}$



Now that we have the HLLE state, we can go back to the Godunov algorithm (15) but we still need the fluxes across the constant state $\underline{U}^{HLLE}$, which we can compute by imposing the conservation of energy and momentum across the nonlinear waves $W_{\rightarrow}$, $W_{\leftarrow}$

These conditions can be defined rather generically and are called the Rankine-Hugoniot conditions. After some algebra, it is possible to show that:

$$\begin{cases} \underline{F}_{L*} = \underline{F}_L + \lambda_L \left(\underline{U}^{HLLE} - \underline{U}_L \right) \\ \underline{F}_{R*} = \underline{F}_R + \lambda_R \left(\underline{U}^{HLLE} - \underline{U}_R \right) \end{cases} \tag{24}$$

(see latex notes)

Using (23) in (24) we obtain

$$\underline{F}_{L*} = \underline{F}_L + \lambda_L \left(\frac{\lambda_R \underline{u}_R - \lambda_L \underline{u}_L + \underline{F}_L - \underline{F}_R}{\lambda_R - \lambda_L} - \underline{u}_L \right)$$

$$= \frac{\underline{F}_L (\lambda_R - \lambda_L) + \lambda_L (\lambda_R \underline{u}_R - \lambda_L \underline{u}_L + \underline{F}_L - \underline{F}_R - \lambda_R \underline{u}_L + \lambda_L \underline{u}_L)}{\lambda_R - \lambda_L}$$

$$= \frac{\lambda_R \underline{F}_L - \lambda_L \underline{F}_R + \lambda_L \lambda_R (\underline{u}_R - \underline{u}_L)}{\lambda_R - \lambda_L}$$

$$\underline{F}_{R*} = \underline{F}_R + \lambda_R \left(\frac{\lambda_R \underline{u}_R - \lambda_L \underline{u}_L + \underline{F}_L - \underline{F}_R}{\lambda_R - \lambda_L} - \underline{u}_R \right)$$

$$= \frac{\underline{F}_R (\lambda_R - \lambda_L) + \lambda_R (\lambda_R \underline{u}_R - \lambda_L \underline{u}_L + \underline{F}_L - \underline{F}_R - \lambda_R \underline{u}_R + \lambda_L \underline{u}_R)}{\lambda_R - \lambda_L}$$

$$= \frac{\lambda_R \underline{F}_L - \lambda_L \underline{F}_R + \lambda_L \lambda_R (\underline{u}_R - \underline{u}_L)}{\lambda_R - \lambda_L}$$

that is

$$\boxed{\underline{F}_{L*} = \underline{F}_{R*} =: \underline{F}_*}$$

the fluxes to be used in the Godunov scheme for the HLLE approximate Riemann solver are therefore:

$$\underline{F}^{HLLE} = \begin{cases} \underline{F}_L & \text{if} \quad x/t < \lambda_L \\ \underline{F}_* & \text{if} \quad \lambda_L < x/t < \lambda_R \\ \underline{F}_R & \text{if} \quad x/t > \lambda_R \end{cases}$$

What we still need are the speeds λ_L, λ_R which are the smallest and the largest of the characteristics, ie

$$\lambda_L = \min\left(0, \lambda_-(\underline{U}_L), \lambda_-(\underline{U}_R)\right)$$

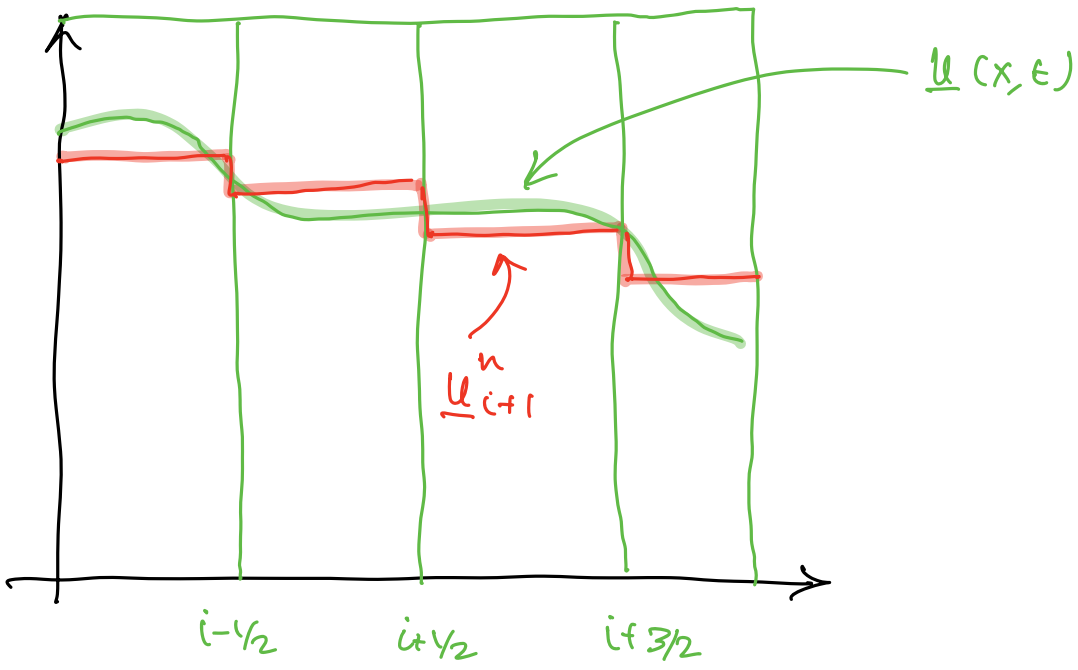
$$\lambda_R = \max\left(0, \lambda_+(\underline{U}_L), \lambda_+(\underline{U}_R)\right)$$

where λ_-, λ_+ are the minimum and maximum eigenvalues of the states $\underline{U}_L$, $\underline{U}_R$. In the case of the equations of classical hydrodynamics

$$\lambda_\pm = v_\pm + c_s$$

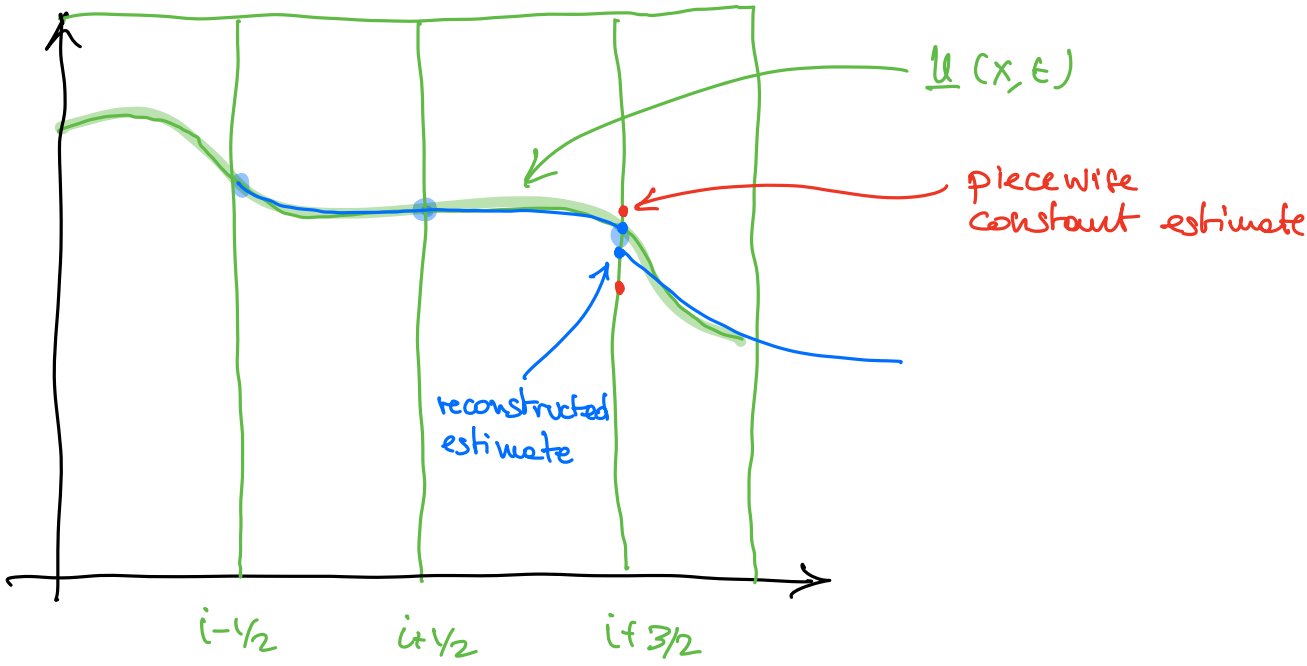
HRSC (High-Resolution Shock-Capturing) Schemes.

We have seen that a Godunov scheme which approximates the solution to a series of piecewise constant states is only first-order accurate and this is clearly something not optimal. The low-order accuracy is essentially the result of the averaging over the cell, which clearly removes any information on the properties of the solution within the cell.



Clearly, the way to improve this and to obtain "high-order" schemes is to restore in some way the information on the behaviour of the function on the basis of the knowledge of the function in neighbouring cells.

This process is called "reconstruction" and essentially amounts to find interpolating functions of various order from which to obtain an improved value of $\underline{U}_{i+1}$



There are many of these reconstruction schemes and a popular one is the piecewise parabolic method (PPM) which provides third-order accuracy in smooth parts of the solution (if solution is discontinuous the reconstruction will be first-order).

Godunov is also behind another important theorem, namely:

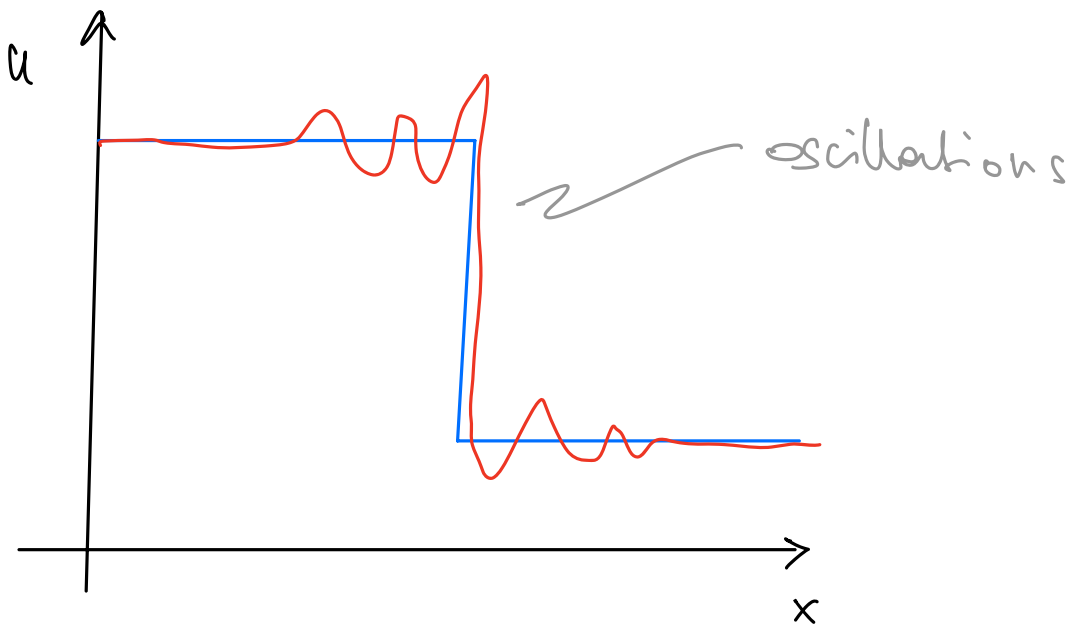
Theorem:

"A linear (i.e., with constant coefficients) and monotonicity preserving scheme is at most first-order accurate".

This theorem essentially states that high-order linear schemes and the absence of oscillations are incompatible requirements.

In other words, high-order accuracy also normally comes with oscillations in the solution.

As a result, oscillation will appear when approximating a discontinuity with a high-order method:



To counter this, methods are devised that are "Total Variation Diminishing" or TVD methods.

Mathematically, if the total variation is

$$TV(\underline{u}^n) := \sum_{i=-\infty}^{+\infty} |\underline{u}_i^n - u_{i+1}^n|$$

then TVD methods are such that the variation does not grow in time, i.e.,

$$TV(\underline{u}^{n+1}) \leq TV(\underline{u}^n).$$

To achieve this, rather sophisticated approaches are necessary. TVD methods are at most second-order and higher-order methods are ENO (Essentially Non Oscillatory).

In summary: HRSC methods are such that

- at least second-order in smooth parts
- sharp resolution of discontinuities
- absence of spurious oscillations
- converge to true solution