Blended Intensive Programme (BIP): Relativistic Fluid Dynamics

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Lecture I

1. Let $\boldsymbol{n}$ be the unit timelike normal to Σ_t and $\boldsymbol{\gamma}$, $\boldsymbol{N}$ are respectively the the projector operator orthogonal to Σ_t and along $\boldsymbol{n}$, *i.e.*, $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and $N^\mu_{\ \nu} = -n^\mu n_\nu$. Show that it is possible to split a covariant tensor $\boldsymbol{W}$ of rank 2 by applying the projection tensor separately on each component of the tensor to obtain
$$W_{\mu\nu} = A n_\mu n_\nu + B_\mu n_\nu + n_\mu C_\nu + Z_{\mu\nu}\,, \tag{1}$$
where
$$A \coloneqq W_{\mu\nu} N^{\mu\nu} = W_{\mu\nu} n^\mu n^\nu\,, \qquad B_\mu \coloneqq -\gamma^\alpha_{\ \mu} W_{\alpha\beta} n^\beta\,, \tag{2}$$
$$C_\nu \coloneqq -\gamma^\alpha_{\ \nu} W_{\beta\alpha} n^\beta\,, \qquad Z_{\mu\nu} \coloneqq \gamma^\alpha_{\ \mu}\gamma^\beta_{\ \nu} W_{\alpha\beta}\,. \tag{3}$$
Further show that the decomposition (1) can be written in the so-called *irreducible* form as
$$W_{\mu\nu} = A n_\mu n_\nu + B_\mu n_\nu + n_\mu C_\nu + \frac{1}{3} W_{\alpha\beta} h^{\alpha\beta}\gamma_{\mu\nu} + W_{\langle\mu\nu\rangle} + \gamma^\alpha_{\ \mu}\gamma^\beta_{\ \nu} W_{[\alpha\beta]}\,. \tag{4}$$
where $W_{\langle\mu\nu\rangle}$ is the trace-free, symmetric[1] and spatial part of the tensor $\boldsymbol{W}$, namely:
$$W_{\langle\mu\nu\rangle} \coloneqq \gamma_\mu^{\ \alpha}\gamma_\nu^{\ \kappa} W_{(\alpha\kappa)} - \frac{1}{3} W_{\alpha\kappa} h^{\alpha\kappa}\gamma_{\mu\nu}\,. \tag{6}$$

2. Prove that if $\boldsymbol{u}$ is a timelike unit four-velocity (*i.e.*, $u^\mu u_\mu = -1$), the covariant three-velocity defined as
$$v^i = -\frac{\gamma^i_{\ \mu} u^\mu}{n_\mu u^\mu}\,, \tag{7}$$
has components given by
$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)\,. \tag{8}$$

3. Prove that the quantity $W \coloneqq \alpha u^t$ is the Lorentz factor since it satisfies the identity
$$W = \left(1 - v^i v_i\right)^{-1/2}\,. \tag{9}$$
Compare expression (8) with the equivalent expression in special relativity.

[1] I recall that it is possible to construct a symmetric or antisymmetric tensor from an arbitrary one, *i.e.*,
$$Z_{(\mu\nu)} \coloneqq \frac{1}{2}\left(Z_{\mu\nu} + Z_{\nu\mu}\right)\,, \qquad Z_{[\mu\nu]} \coloneqq \frac{1}{2}\left(Z_{\mu\nu} - Z_{\nu\mu}\right)\,, \tag{5}$$
and that an arbitrary tensor can always be decomposed into its symmetric and antisymmetric parts, *i.e.*, $Z_{\mu\nu} = Z_{(\mu\nu)} + Z_{[\mu\nu]}$.

Lecture II

1. Assuming for simplicity that the flow is one-dimensional (*i.e.*, for $\mu = 0, 1$) and the spacetime flat, we rewrite the conservation equations for energy and linear momentum

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{10}$$

can be written in a Cartesian coordinate system as

$$\partial_t \left[\left(e + pv^2\right) W^2 \right] + \partial_x \left[\left(e + p\right) W^2 v \right] = 0 \,, \tag{11}$$

$$\partial_t \left[\left(e + p\right) W^2 v \right] + \partial_x \left[\left(ev^2 + p\right) W^2 \right] = 0 \,, \tag{12}$$

where $u^\mu = W(1, v)$ and $W = (1 - v^2)^{-1/2}$ is the Lorentz factor.

2. Linearize Eqs. (11)–(12) by introducing perturbations of the type

$$e = e_0 + \delta e \,, \qquad p = p_0 + \delta p \,, \qquad v = v_0 + \delta v = \delta v \,, \tag{13}$$

Show that the resulting equations satisfy a wave equation

$$\Box \delta e = 0 \,. \tag{14}$$

What are the assumptions needed to derive Eq. (14)? What is the speed of propagation of these waves?

3. Show that the following definitions of energy-momentum tensor are equivalent

$$T^{\mu\nu} = (e + p) u^\mu u^\nu + p g^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu} \,,$$
$$T^{\mu\nu} = E n^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu} \,.$$

where E, S^μ and $S^{\mu\nu}$ are the Eulerian energy density, the momentum density and the purely spatial energy-momentum tensor, respectively. Show also that the following definitions are possible for these quantities

$$S^{\mu\nu} = \rho h W^2 v^\mu v^\nu + p \gamma^{\mu\nu} \,,$$
$$S^\mu = \rho h W^2 v^\mu \,,$$
$$E = \rho h W^2 - p \,.$$

4. The conservation of energy and momentum $\nabla_\mu T^{\mu\nu} = 0$ can be written explicitly as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^\mu_{\ \nu}) = \frac{1}{2} T^{\mu\lambda} \partial_\nu g_{\mu\lambda} \,.$$

Concentrating to the spatial $\nu = j$, show that the following identity holds when considering the left-hand side of the above equation

$$\partial_\mu (\sqrt{-g} T^\mu_{\ j}) = \partial_t (\sqrt{\gamma} S_j) + \partial_i [\sqrt{\gamma} (\alpha S^i_{\ j} - \beta^i S_j)] \,.$$

Similarly, show that the right-hand side satisfies the following identity

$$\frac{1}{2} \sqrt{-g} T^{\mu\nu} \partial_j g_{\mu\nu} = \sqrt{-g} \left(\frac{1}{2} S^{ik} \partial_j \gamma_{ik} + \frac{1}{\alpha} S_i \partial_j \beta^i - E \, \partial_j \ln \alpha \right) \,,$$

5. The conservation of the energy density $n_\nu \nabla_\mu T^{\mu\nu} = 0$ can be written explicitly as

$$\nabla_\mu (T^{\mu\nu} n_\nu) = T^{\mu\nu} \nabla_\mu n_\nu .$$

Show that the following identity holds when considering the left-hand side of the conservative formulation of the energy-density equation

$$-\sqrt{-g} \nabla_\mu (T^{\mu\nu} n_\nu) = \partial_t (\sqrt{\gamma} E) + \partial_i [\sqrt{\gamma} (\alpha S^i - \beta^i E)] \, .$$

Similarly, show that the right-hand side satisfies the following identity

$$-\sqrt{-g} T^{\mu\nu} \nabla_\mu n_\nu = \sqrt{-g} \left(K_{ij} S^{ij} - S^i \partial_i \ln \alpha \right) \, .$$

1 Lecture III

Hyperbolic PDEs: nonlinear case in 1+1 D

1. Show that the second-order accurate finite-difference representation of the second spatial derivative is given by

$$\partial_x^2 u\Big|_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)\,.$$

   What is the expression for the third derivative $\partial_x^3 u\Big|_j^n$? What is the accuracy order of your expression?

2. Consider the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,, \tag{15}$$

   on a grid with extents $x \in [0, 1]$ and with initial data $u(x, 0) = 1$ for $x < 1/2$ and $u(x, 0) = 0$ for $x \geq 1/2$, namely a step function.

   (a) Build a numerical code to solve Eq. (15) using methods such as Lax-Friedriechs and Lax-Wendroff.

   (b) Build a numerical code to solve Eq. (15) using methods the upwind method and a conservative formulation.

3. **Optional:**

   (a) Build a numerical code to solve Eq. (15) using HRSC methods, i.e., using the HLLE approximate Riemann solver.

   (b) Build a numerical code using the Lax-Friedrichs scheme to solve Eq. (15) with initial data given $u(x, 0) = \sin(2\pi x)$ and continue the evolution after a discontinuity has been produced. What happens if you use an HRSC method?

   Hint:

   - Do not forget the importance of a conservative formulation.