BIP, Timisoara May 2026

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Lecture Ia: 3+1 split in general relativity

Lecture Ib: ADM formulation

Lecture IIa: Relativistic Hydrodynamics

Lecture IIb: Numerical Methods for Hyperbolic PDEs

Lecture IIIa: Application to BNSs: bulk dynamics & GWs

Lecture IIIb: Application to BNSs: EM emission and neutrinos

Extra material: conformal formulations and gauges

Lecture Ia: 3+1 split in GR

The theory of general relativity is as fascinating as it is hard to solve. With the exception of a few solutions with high degree of symmetry, most of which we have actually discussed, the solution of the Einstein equations cannot be performed analytically.

Note that such a solution requires finding

1. a metric tensor sourced by a specific distribution of matter/energy ($T_{\mu\nu}$):

$$\boxed{g_{\mu\nu}\big|_{t=0} \rightleftarrows R_{\mu\nu}\big|_{t=0}}$$ initial data

2. determine the evolution of such a metric tensor

$$\boxed{\partial_t g_{\mu\nu} = \partial_x^2 \cdots \longleftrightarrow R_{\mu\nu}\big|_{t\neq 0}}$$ "fields evolution"

3. determine the evolution of the energy-momentum tensor in the "new" background spacetime

$$\boxed{\partial_t e = f(g_{\mu\nu}); \quad \partial_t v^i = g(g_{\mu\nu});}$$ "matter" evolution

4. repeat 1.–3. till stationarity or staticity is found while coping with any singularity (physical, coordinate, matter) that may be produced during the evolution.

All of this needs to be computed in spacetimes without symmetries (ie 3 spatial dimensions) and covering large range in scales: typical wavelengths of gravitational radiation are much larger than size of sources $\lambda_{GW} \gg R$

The sheer number of equations to be solved then forces the use of supercomputers and hence a new layer of complexity.

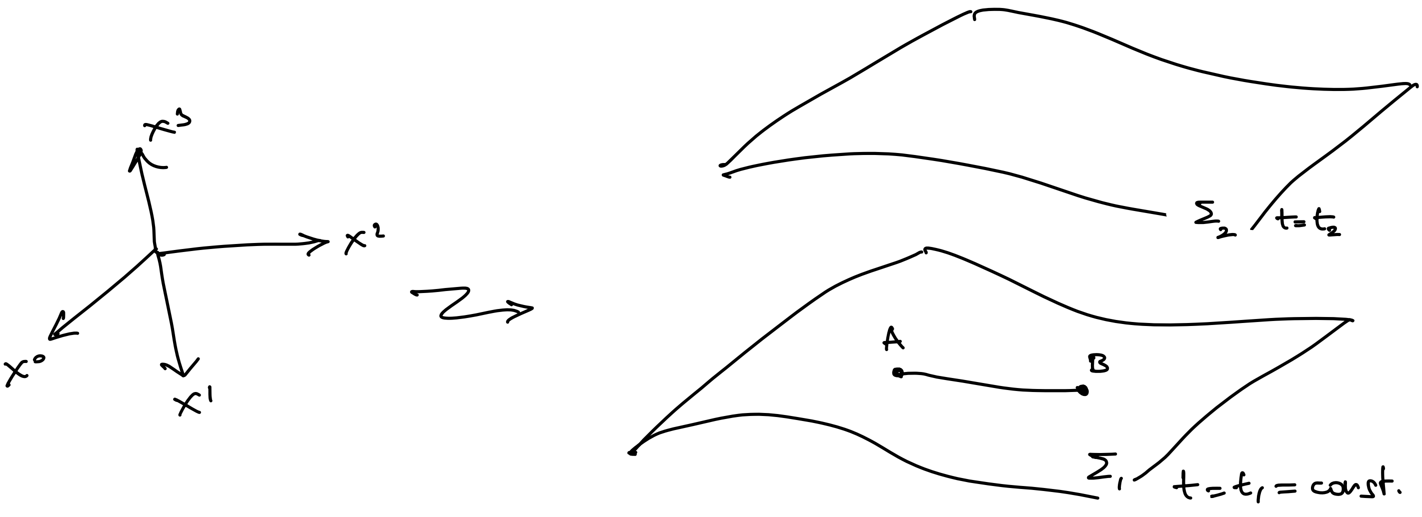
All of this is part of "numerical gravity" or numerical relativity, a very active area of modern research that includes: suitable formulations of the Einstein equations, matter equations, gauge conditions, initial data, numerical methods.

The amount of information to be discussed would require a whole course and indeed there are books dedicated to this only; here we will just cover the basics but enough to get you started!

3+1 splitting of spacetime

Let $\mathcal{M}$ be a four-dimensional manifold endowed with a metric $\underline{g}$. One of the first concepts introduced in general relativity is that space and time are equivalent coordinates and there is nothing special about time.

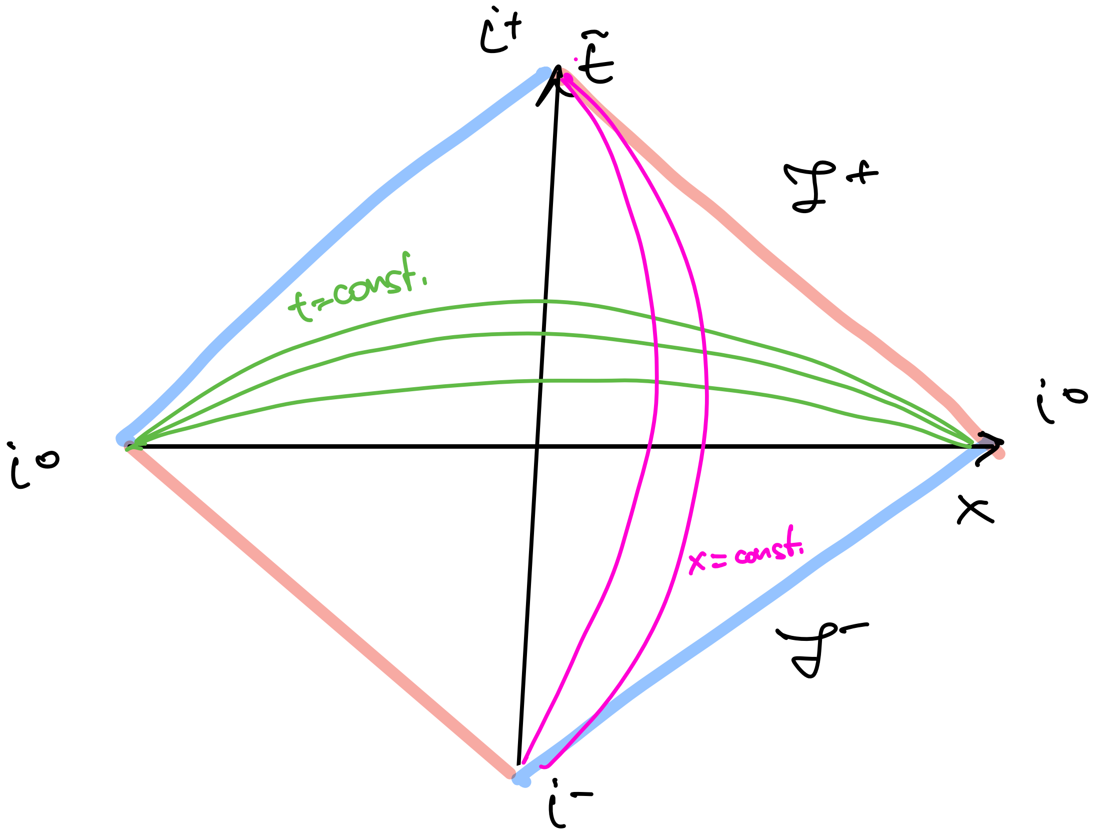
However, physical processes have a clear time direction set by thermodynamics and so it can be useful to split $\mathcal{M}$ in hypersurfaces Σ where the coordinate t is constant



Events A and B are spacelike separated because they live on the same $t=$const hypersurface.

Note that this splitting in 3+1 is not the only one possible and not necessarily the best. To see this, let's go back to the conformal description of a Minkowski spacetime; in 1+1 dimensions we have considered x-const and t-const slices.

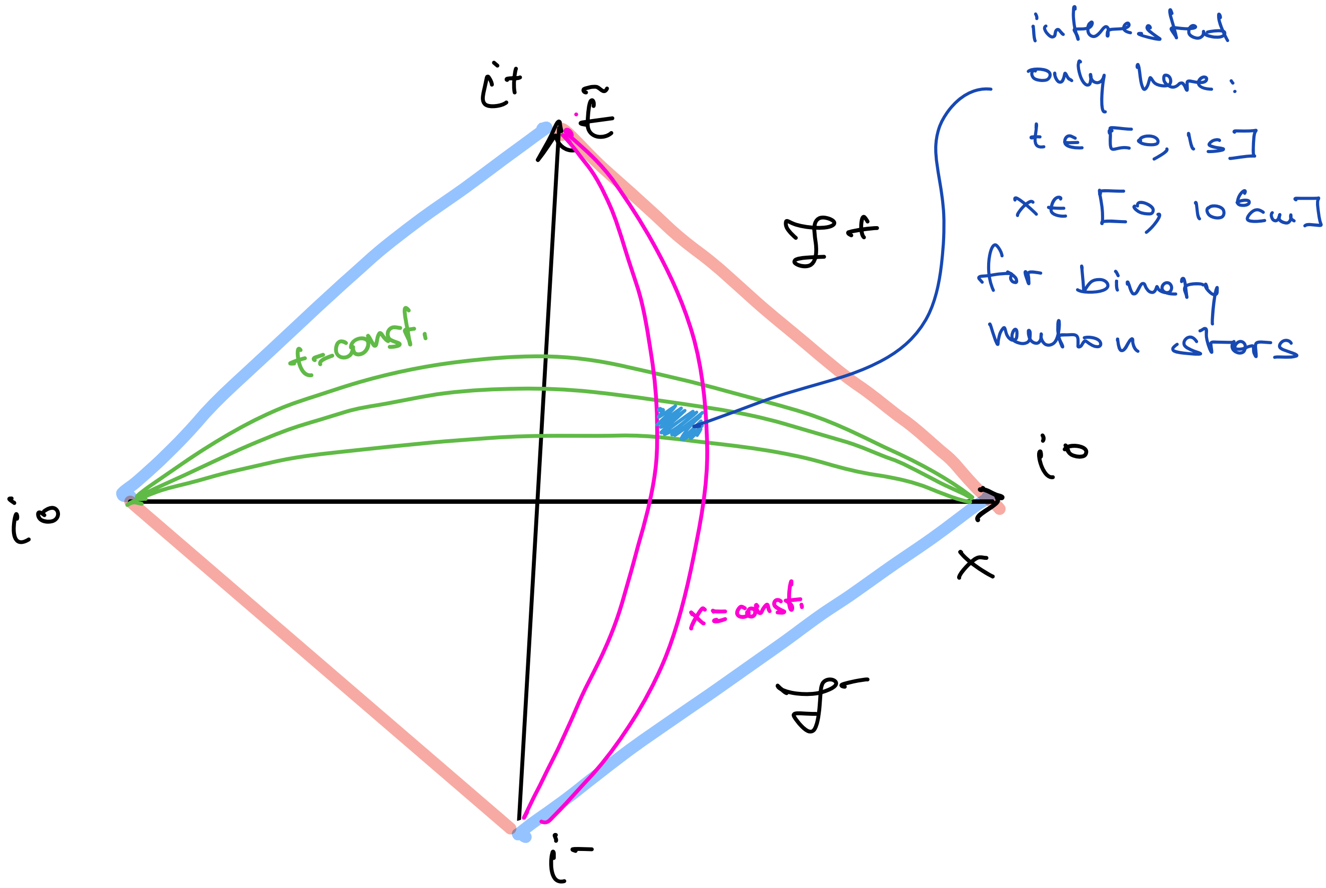
3+1 decomposition



Normally, in numerical simulations, either in numerical relativity or in other areas of science, we are not interested in covering the whole spacetime, ie having

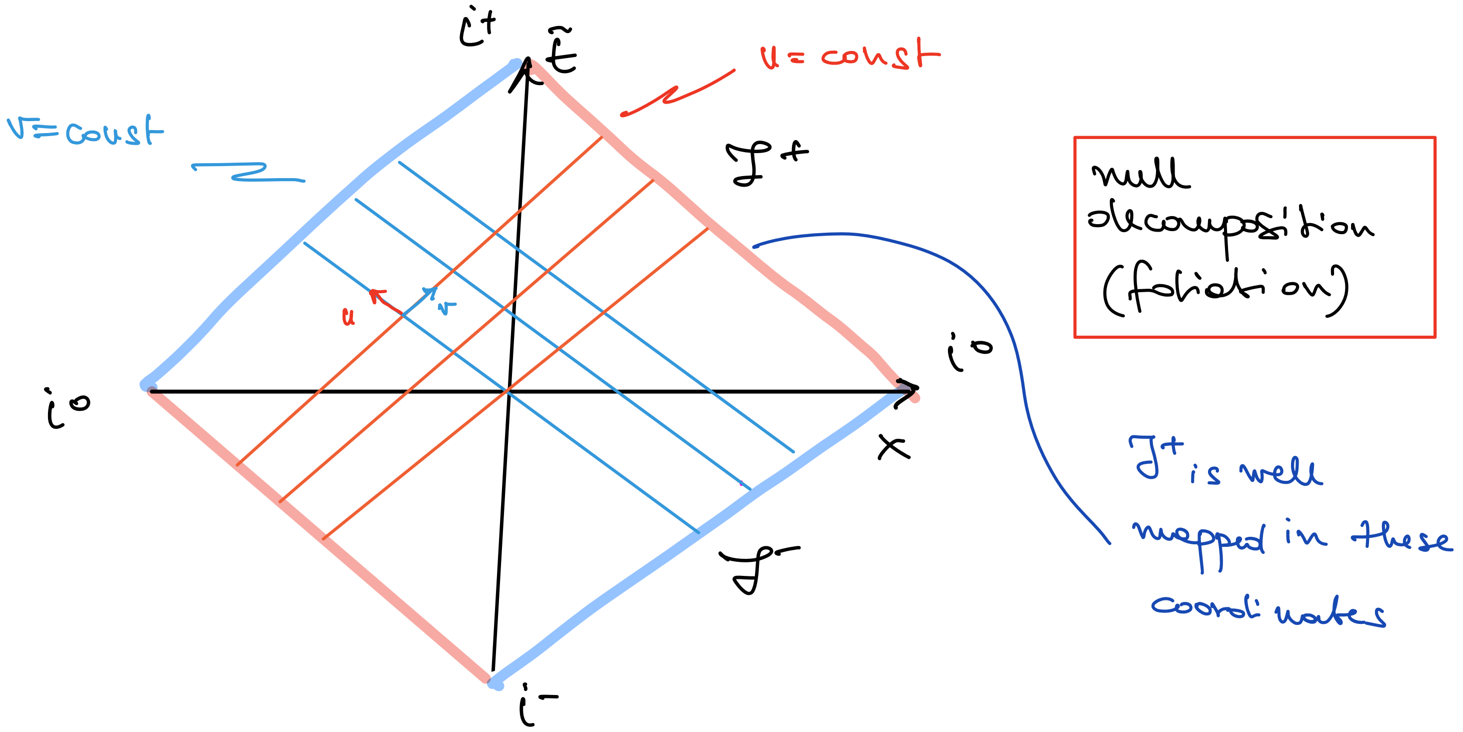
$$x \to i^0 \quad \text{or} \quad t \to i^+$$

On the contrary, we are interested in obtaining a solution only in a rather small patch of the spacetime

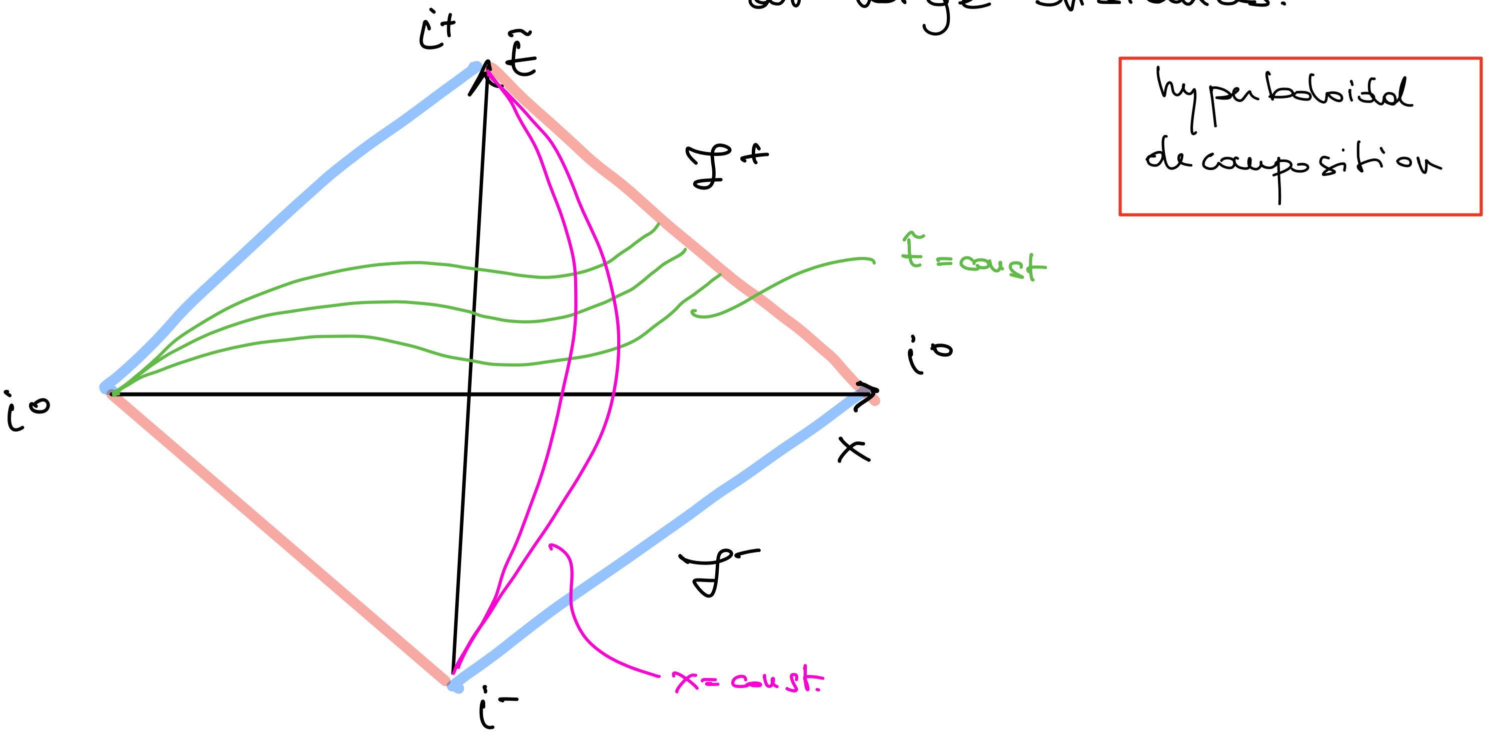


However, in gravity we are particularly interested in the radiation content of the spacetime and to this scope, we could have introduce a different set of "null" coordinates u, v such that

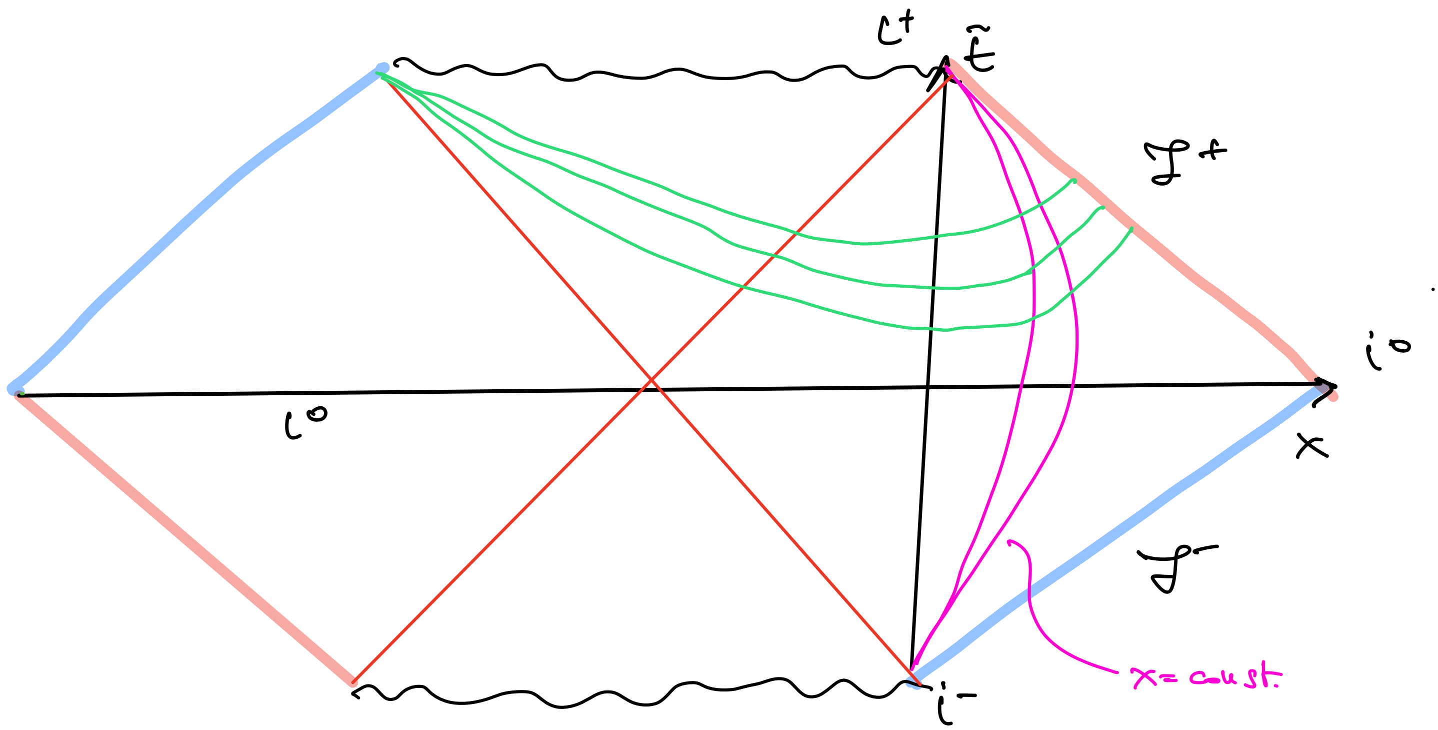
$$\begin{cases} u := t + r \\ v := t - r \end{cases} \iff \begin{cases} t = \frac{1}{2}(u + v) \\ r = \frac{1}{2}(u - v) \end{cases}$$



A null foliation is much better to study asymptotic properties of radiation: light cones are not distorted. At the same time, the initial data is harder to compute. A possible compromise is a "hyperboloidal" decomposition, which is both spacelike and null at large distances.



The choice of proper coordinates, or gauges, becomes even more severe if a singularity is present in the spacetime for, in that case, one needs to lay coordinates that reach it only asymptotically. For example, via singularity avoiding slicings (e.g. "puncture").



Hereafter, we will consider a 3+1 decomposition as it is the most common and convenient to use when adopting (novel) numerical methods; this is because numerical analysts still adopt Galilean concept of time.

Given an (hyper) surface Σ, the first quantity we can define is its local normal vector $\underline{n}$

Since $t = $ const. on Σ, this four-vector will have to be oriented along the direction in which the time coordinate varies. This can be defined via a one-form (t is a scalar function) $\tilde{\Omega}$ such that

four vector $\begin{pmatrix}1\\0\end{pmatrix}$

$$\boxed{\Omega_\mu := \nabla_\mu t} \qquad (1)$$

and clearly the one-form associated to $\underline{n}$, i.e., $\tilde{n}$ will have to be proportional to $\tilde{\Omega}$.

We can therefore set

$\begin{pmatrix}0\\1\end{pmatrix}$ one-form

proportionality constant

$$\boxed{n_\mu = A\,\Omega_\mu} \qquad (2)$$

and compute

$$|\Omega|^2 = \Omega_\mu \Omega^\mu = g^{\mu\nu}\Omega_\mu \Omega_\nu$$

$$= g^{\mu\nu}\nabla_\mu t \, \nabla_\nu t$$

$$= g^{tt}\nabla_t t \, \nabla_t t = g^{tt} \qquad (3)$$

This allows us to compute the norm of $\tilde{n}$:

$$|n|^2 = n_\mu n^\mu = A^2 \Omega_\mu \Omega^\mu = A^2 g^{tt} = -1 \qquad (4)$$

where the last equality reflects the fact that we want $\underline{n}$ to be a unit time-like normal vector

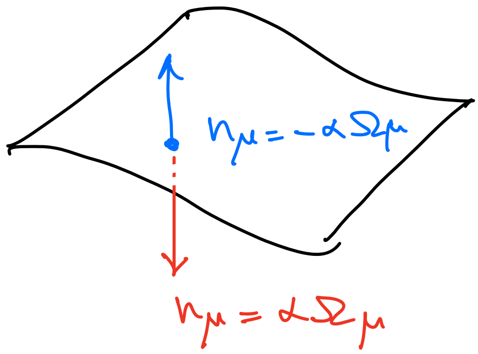
Using Eq. (4) we deduce that

$$\boxed{A^2 = -\frac{1}{g^{tt}} =: \alpha^2} \tag{5}$$

so that $A = \begin{cases} -\alpha & : \text{future oriented} \\ +\alpha & : \text{past oriented} \end{cases}$

where the different signs mark the direction in which the norm is pointing.

$$n_\mu = A\Omega_\mu = -\alpha\,\Omega_\mu = -\alpha\,\nabla_\mu t \tag{6}$$



$$\boxed{n_\mu = (-\alpha, 0, 0, 0)} \; . \tag{7}$$

purely time-like vector; norm of $\underline{n}$ depends on position ($\alpha = \alpha(x^i)$)

Using $\underline{g}$ we can compute also the components of $\underline{n}$:

$$n^\mu = g^{\mu\nu} n_\nu = -\alpha\, g^{\mu\nu}\nabla_\nu t = -\alpha\,\nabla^\mu t \tag{8}$$

Note that $n^i = g^{i0} n_0$ is not necessarily zero!

Since we have the full four-metric $\underline{g}$ and the normal to Σ_t $\underline{n}$, we can introduce a rank-2 tensor that is orthogonal to $\underline{n}$ and that will therefore represent the metric on Σ_t:

$$\boxed{\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu} \tag{9}$$

AB

where clearly :

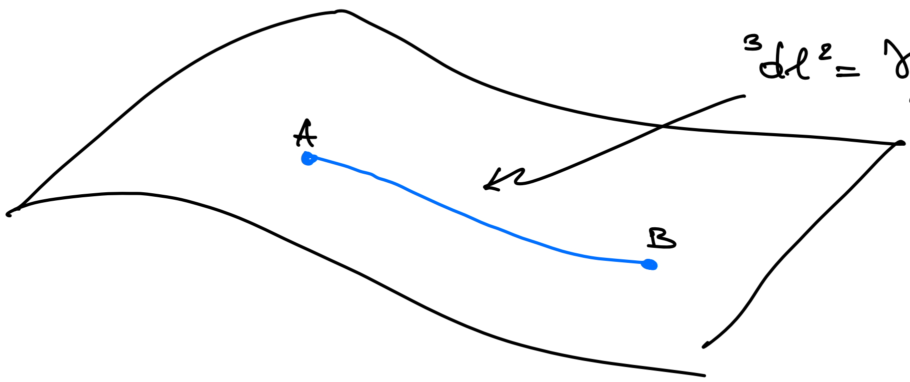
(10) $\boxed{\underline{\gamma} \cdot \underline{n} = 0}$ $\qquad \gamma_{\mu\nu} n^{\mu} = (g_{\mu\nu} + n_{\mu} n_{\nu}) n^{\mu} =$

$$= n_{\nu} - n_{\nu} = 0$$

$\boxed{\underline{\gamma} \Leftrightarrow \underline{g}\big|_{\Sigma}}$: $\underline{\gamma}$ is the metric restricted to the slice Σ_t and hence measures distances between events at the same time coordinate

$\underline{\gamma}$ is a 4-dimensional spacetime object but the only nonzero part is in the spatial part :

$$\gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{ij} \end{pmatrix} \qquad (11)$$



The mixed components of $\underline{\gamma}$ are simply

$$\boxed{\gamma^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu} n_{\nu}} \qquad (12)$$

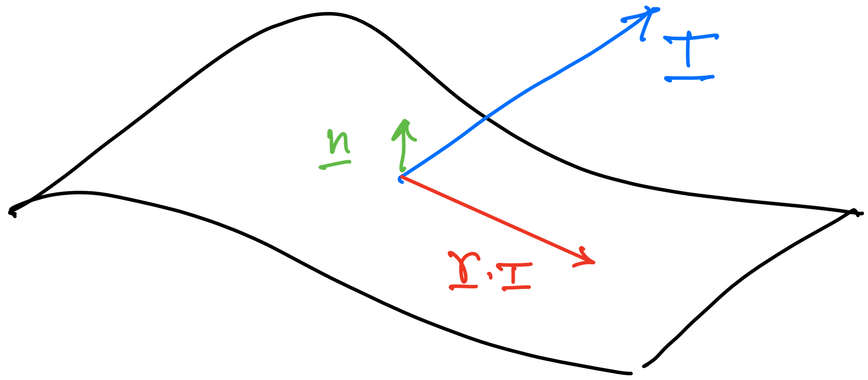
so that

$$\gamma^{\mu\nu} A_\mu = A^\nu \qquad \text{if} \qquad A_\mu n^\mu = 0 \qquad (13)$$

$$\left(\gamma^{\mu\nu} A_\mu = (g^{\mu\nu} + n^\mu n^\nu) A_\mu = g^{\mu\nu} A_\mu + n^\mu n^\nu A_\mu = A^\mu \right)$$

In other words, $\underline{\gamma}$ can be used to raise/lower indices but only of tensors that are fully spatial, that is, that live only on Σ_t (3D). For generic 4D tensors the raising/lowering of the indices must be done with the full metric $\underline{g}$.

In this sense $\underline{\gamma}$ is a spatial projection tensor: given a tensor $\underline{T}$, $\underline{\gamma} \cdot \underline{T} = \underline{W}$ where $\underline{W}$ is the spatial projection of $\underline{T}$ on Σ_t.



In a similar way, we can construct a rank-2 tensor that projects "out of Σ_t"; this is simply given by

$$(14) \qquad \boxed{N^\mu{}_\nu := -\, n^\mu n_\nu}$$

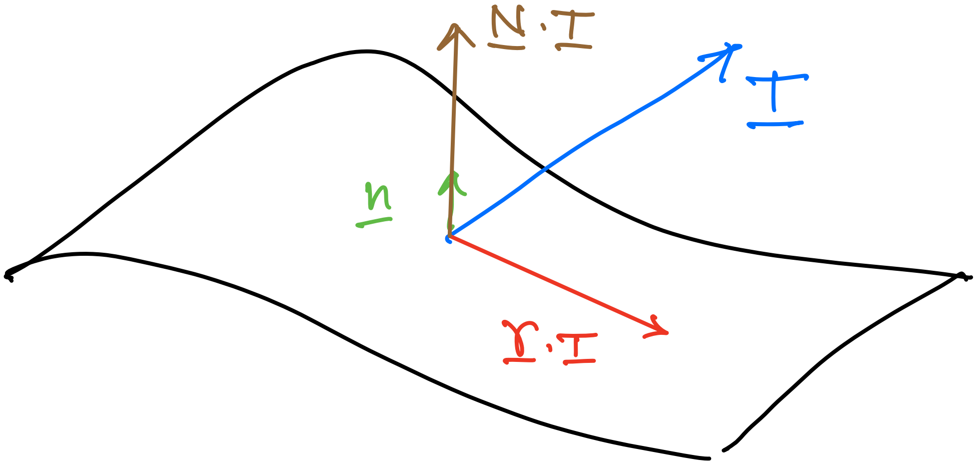
where $\boxed{\underline{N} \cdot \underline{\gamma} = 0}$

$$N^{\mu}{}_{\nu}\, \gamma^{\nu}{}_{\mu} = -(n^{\mu} n_{\nu})(\delta^{\nu}{}_{\mu} + n^{\nu} n_{\mu})$$
$$= -n^{\mu} n_{\mu} - n^{\mu} n_{\mu} n^{\nu} n_{\nu}$$
$$= 1 - 1 = 0 \quad \checkmark$$

obviously, also $\underline{n} \cdot \underline{\gamma} = 0$

$$n^{\mu} \gamma^{\nu}{}_{\mu} = n^{\mu} (\delta^{\nu}{}_{\mu} + n^{\nu} n_{\mu}) = n^{\nu} - n^{\nu} = 0$$

In this way, ie with the introduction of $\underline{\gamma}$ and $\underline{N}$ we have constructed a framework that allows us to decompose (split) any tensor $\underline{T}$ into a purely spatial $(\underline{\gamma} \cdot \underline{T})$ and in a purely time $(\underline{N} \cdot \underline{T})$ part.



In turn, given a tensor $\underline{T}$, we can always decompose into a purely spatial and purely time part:

$$\underline{T} = \underline{\gamma} \cdot \underline{T} + \underline{N} \cdot \underline{T} = (\underline{\gamma} + \underline{N})\, \underline{T} = \underline{A} + \underline{B}$$

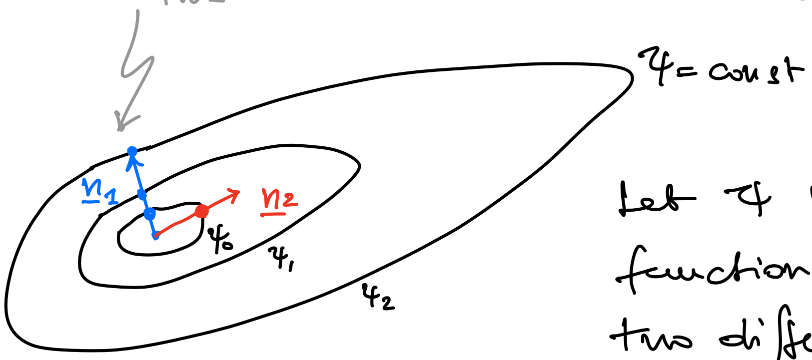
Note

$\underline{n}$ is the timelike unit normal to any point on Σ_t but it does not necessarily point in the same direction in which the time coordinate changes. This is because when moving from one slice to the next also the spatial coordinates can change and this creates a spatial "shift".

To understand this better, we need to take a step back and recall that a one-form represents the gradient of a scalar function, say τ, and that to turn the one-form into a scalar (a number) we need to project such a gradient in a given direction; a gradient per-se is not sufficient and requires the specification of a direction along which to consider the gradient.

The contraction of the one-form (gradient) with a vector (direction) will provide a number expressing how rapidly the scalar function varies along that direction.

this could be the altitude in a geographical map



Let Ψ be a scalar function and $\underline{n}_1$, $\underline{n}_2$ two different vectors of the same length (e.g., unit).

Let $\tilde{\Omega} := \nabla \Psi$; $\Omega_\mu := \nabla_\mu \Psi$

$$\underline{n} \cdot \tilde{\Omega} = x \in \mathbb{R}$$

Given a scalar function $\Psi = \Psi(x^\mu)$, the real-valued function $x = x(x^\mu) := \underline{n} \cdot \tilde{\Omega}$ will be different for different directions $\underline{n}$. In the example above

$\underline{n}_1 \cdot \tilde{\Omega} = 3$; $\underline{n}_2 \cdot \tilde{\Omega} = 1$: this is the number of iso-levels crossed by $\underline{n}_1$ and $\underline{n}_2$.

(these could be the number of steps taken along $\underline{n}_1$ and $\underline{n}_2$)

In this sense, the transition from one iso-level to the next will not be possible having the same normal vector $\underline{n}$ but will be a function of position.

Stated differently, if we want to make sure that the transition from one level to the next is the same, that is:

$$\boxed{\underline{n} \cdot \tilde{\Omega} = \text{const}} \tag{15}$$

then the length of the vector $\underline{n}$ will be different in different positions as $\underline{n}$ will have to adapt to a locally changing value of $\tilde{\Omega} = \nabla \Psi$.

Using the analogy of the altitude in a geographical map, I would need to take more steps in the NE direction ($\underline{n}_2$) than in the SW direction to reach the same altitude Ψ_2

If we take $\underline{n}$ to be the unit time-like normal to Σ_t and Ω the gradient of the time coordinate, then

$$\underline{n} \cdot \tilde{\Omega} = n^\mu \Omega_\mu \overset{n_\mu = -\alpha\Omega_\mu}{=} -\frac{1}{\alpha} n^\mu n_\mu \overset{n^\mu n_\mu = -1}{=} \frac{1}{\alpha} \neq \text{const.} \tag{16}$$

since $\alpha = \alpha(x^\mu)$.

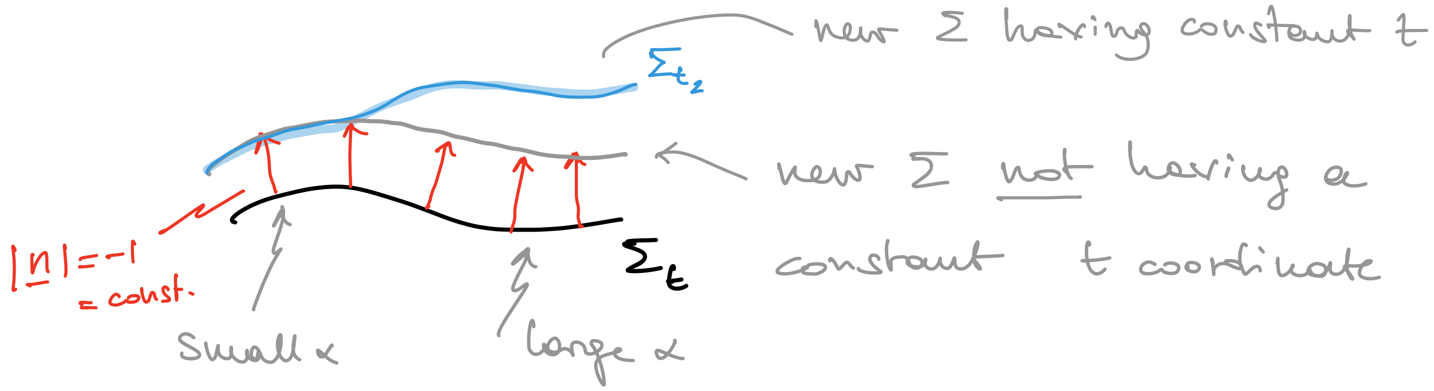
In other words, eq. (16) states that time progresses at different rates at different positions on Σ_t; this is not surprising at all!

Furthermore, since $\underline{n} \cdot \tilde{\Omega} = \alpha^{-1}$, the function α will describe how time varies differently on the slice Σ_t.

Since $\underline{n}\cdot\tilde{\Omega} \neq$ const we need a new time-like vector $\underline{t}$, $\underline{t}\cdot\underline{t} < -1$ such that

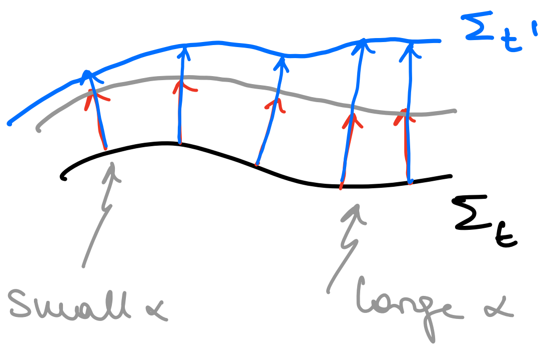
$$\boxed{\underline{t}\cdot\tilde{\Omega} = 1} \qquad (17)$$

This new time-like vector would still point out of Σ_t but it would guarantee that when moving from Σ_t to another spacelike hypersurface, all points will be moved so that they will all have the same $t=$ const coordinate



Satisfying (17) is trivial since all we need it to correct $\underline{n}$ by α:

$$(18) \quad \boxed{\underline{t} = \alpha\,\underline{n}} \qquad \Longrightarrow \quad t^\mu t_\mu = \alpha^2 n^\mu n_\mu = -\alpha^2$$



on this hypersurface all points in Σ will be dragged by the correct amount of time to be again on a constant $t=$ const slice.

AIS

At the same time, we need to take into account that the spatial position may also change and so we need to correct for it; as a result the most general definition of the time-progressing time-like vector is

$$\underline{t} = \alpha \underline{n} + \underline{\beta} := \underline{e}_t \qquad (19)$$

where $\underline{\beta} \cdot \underline{n} = 0$, ie $\underline{\beta}$ is purely spatial. As a result $\underline{t} \cdot \tilde{\Omega} = (\alpha \underline{n} + \underline{\beta}) \tilde{\Omega} = \alpha \underline{n} \cdot \tilde{\Omega} + \underline{\beta} \cdot \tilde{\Omega}$ (struck out)

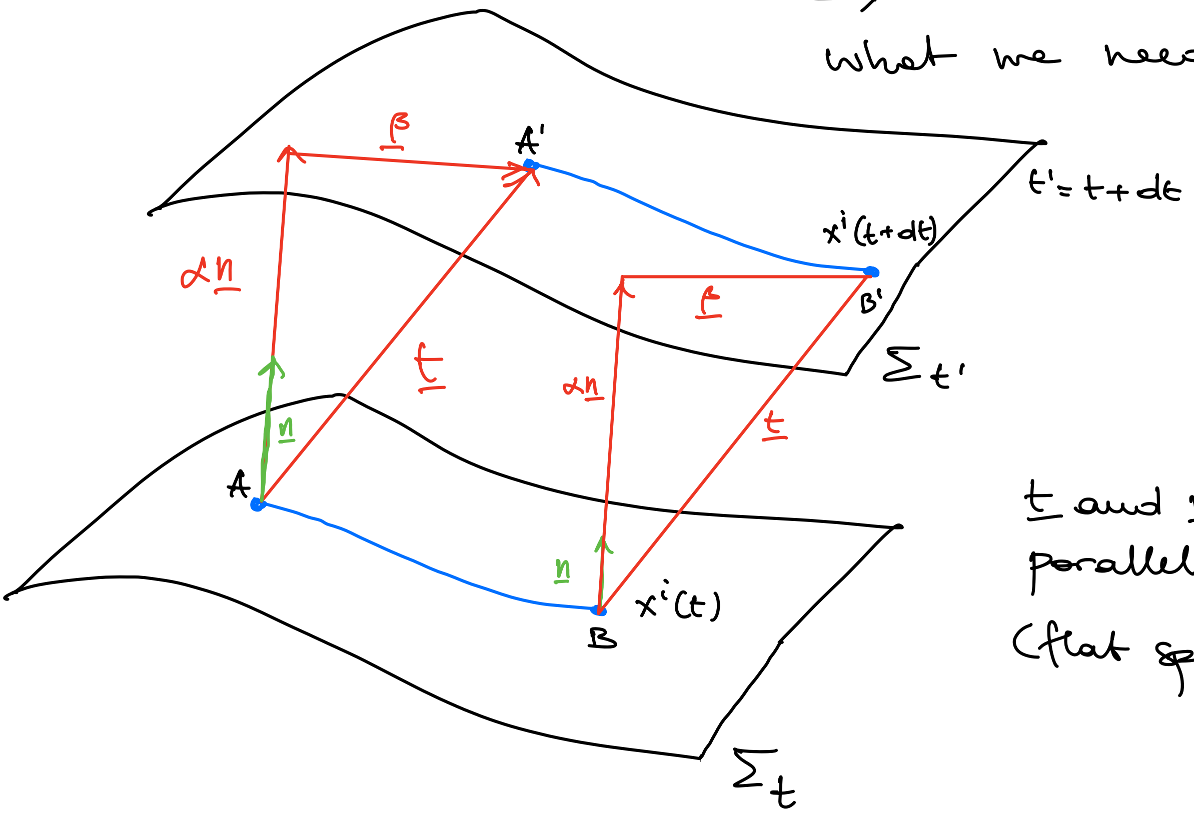
$\underline{n} \cdot \tilde{\Omega} = n^\mu \Omega_\mu = n^0 = g^\mu$ ($\Omega_0 = 1$, $g^{00} = -1/\alpha^2$)

$= g^{\mu 0} n_\mu = g^{00} n_0 = \frac{1}{\alpha}$ ($n_0 = -\alpha$)

$$= \alpha \underline{n} \cdot \tilde{\Omega} = 1$$

$$= \alpha \cdot \left(-\frac{1}{\alpha^2}\right) \cdot -\alpha = 1 \quad \checkmark \qquad (20)$$

So, the vector $\underline{t}$ does what we need...



$\underline{t}$ and $\underline{n}$ are parallel if $\underline{\beta} = 0$ (flat spacetime)

Obviously, $\underline{t}$ is not a unit timelike vector:

$$\underline{t}\cdot\underline{t} = (\alpha \underline{n} + \underline{\beta})\cdot(\alpha \underline{n} + \underline{\beta}) = \alpha^2 \underline{n}\cdot\underline{n} + 2\alpha \underline{n}\cdot\underline{\beta} + \underline{\beta}\cdot\underline{\beta}$$

$$= -\alpha^2 + \beta^i \beta_i \qquad (\beta^0 = 0)$$

$$\neq -1$$

$\alpha = \alpha(x^\mu)$ is called the "lapse" function

$\underline{\beta}$ is called the "shift vector"

Note:

- $n_\mu = -\alpha \Omega_\mu$;

$$n^\mu = g^{\mu\nu} n_\nu = g^{\mu\nu}(-\alpha)\Omega_\nu = -\alpha \Omega^\mu \Rightarrow$$

$$\underline{n} = -\alpha \underline{\Omega}$$

$\underline{n}$ and $\underline{\Omega}$ are anti-aligned ($\alpha > 0$)

- $$\underline{t} = \alpha \underline{n} + \underline{\beta} \overset{\underline{\beta}=0}{=} \alpha \underline{n} = -\alpha^2 \underline{\Omega}$$

$\underline{t}$ and $\underline{\Omega}$ are also anti-aligned

The splitting of time and spatial coordinates and the introduction of the lapse function and shift vector allow us to write the generic metric $\underline{g}$ in its 3+1 decomposition.

Let's now lay a set of coordinates $\{x^\mu\}$ with corresponding basis vectors $\underline{e}_\mu$ (not necessarily unit vectors). Then we know that the components of the metric tensor will reflect the scalar products of these basis vectors, i.e.,

$$g_{\alpha\beta} = \underline{e}_\alpha \cdot \underline{e}_\beta \implies$$

$$g_{tt} = \underline{e}_t \cdot \underline{e}_t = \underline{t} \cdot \underline{t} = -\alpha^2 + \beta^i \beta_i \quad (21)$$

$$g_{ti} = \underline{e}_t \cdot \underline{e}_i = (\underline{t} \cdot \underline{\gamma})_i = t^\mu \gamma_{\mu i} \quad (22)$$

$$= t^\mu (g_{\mu i} + n_\mu n_i)$$

$$= (\alpha n^\mu + \beta^\mu) g_{\mu i}$$

$$= \alpha n_i + \beta_i = t_i$$

$$g_{ij} = (\underline{\gamma} \cdot \underline{\gamma})_{ij} = \gamma_{ij}$$

As a result, the 3+1 line element will be in full generality:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -(\alpha^2 - \beta^i\beta_i)\,dt^2 + 2\beta_i\, dx^i dt + \gamma_{ij}\, dx^i dx^j \qquad (23)$$

$$g_{\mu\nu} = \begin{pmatrix} -(\alpha^2 - \beta^i\beta_i) & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad (24)$$

and its inverse

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix} \qquad (25)$$

Similarly:

$$n^\mu n_\mu = -1 = g^{\mu\nu} n_\mu n_\nu \overset{n_i = 0}{=} g^{00}(n_0)^2 = (-1/\alpha^2)(n_0)^2 \Rightarrow$$

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (26)$$

$$n^\mu = g^{\mu\nu} n_\mu = g^{\mu 0} n_0 = \frac{1}{\alpha}(1, -\beta^i) \qquad (27)$$

Let's work out some example: Schwarzschild solution in (ingoing) Eddington Finkelstein coordinates

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^{*2} + \frac{4M}{r}\,dt^* dr + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$$

$$\alpha^2 = \left(1+\frac{2M}{r}\right)^{-1}$$

$$\beta^r = \frac{2M/r}{1+2M/r} \; ; \; \beta_r = \frac{2M}{r} \; ; \; \beta^\theta = 0 = \beta^\phi = \beta_\theta = \beta_\phi$$

$$\gamma_{rr} = 1+\frac{2M}{r} \; ; \quad \gamma^{rr} = 1-\frac{2M}{r} + \frac{(2M/r)^2}{1+2M/r} = g^{r\alpha} g^{r\beta} \gamma_{\alpha\beta}$$

$$\gamma_{\theta\theta} = r^2 \; ; \quad \gamma^{\theta\theta} = \frac{1}{r^2} \qquad (28)$$

$$\gamma_{\phi\phi} = r^2 \sin^2\theta \; ; \quad \gamma^{\phi\phi} = \frac{1}{r^2\sin^2\theta}$$

$$\gamma_{r\theta} = 0 = \gamma_{r\phi} = \gamma_{\theta\phi} = \gamma^{r\theta} = \gamma^{r\phi} = \gamma^{\theta\phi}.$$

Similarly, Kerr solution in Kerr-Schild coordinates

$$\alpha^2 = \left(1+\frac{2M}{\Sigma}\right)^{-1}$$

$$\beta^r = \frac{2Mr/\Sigma}{1+2Mr/\Sigma} \; ; \; \beta_r = \frac{2Mr}{\Sigma} \; ; \; \beta^\theta = 0 = \beta_\theta$$

$$(29)$$

$$\beta^\phi = 0 \qquad ; \; \beta_\phi = -\,2Mra\sin^2\theta \,/\, \Sigma$$

$$\gamma_{rr} = 1 + \frac{2Mr}{\Sigma} \; ; \quad \gamma^{rr} = \frac{\Delta}{\Sigma} + \frac{(2Mr/\Sigma)^2}{1+2Mr/\Sigma}$$

$$\gamma_{\theta\theta} = \Sigma \; ; \quad \gamma^{\theta\theta} = \frac{1}{\Sigma}$$

$$\gamma_{\phi\phi} = \frac{A \sin^2\theta}{\Sigma} \; ; \quad \gamma^{\phi\phi} = \frac{1}{\Sigma \sin^2\theta}$$

$$\gamma_{r\phi} = -\left(1 + \frac{2Mr}{\Sigma}\right) a \sin^2\theta \; ; \quad \gamma^{r\phi} = \frac{a}{\Sigma}$$

$$\gamma_{\theta\phi} = 0 = \gamma^{\theta\phi}$$

$$\gamma_{r\theta} = 0 = \gamma^{r\theta}$$

□

These examples are useful to show that the shift vector and hence the possibility that coordinate lines can be distorted with time is not a consequence of the 3+1 splitting but simply a way of capturing these phenomena via a shift vector. Hence, the β^ϕ component in the Kerr metric is related to the Lense-Thirring (frame dragging) angular velocity.

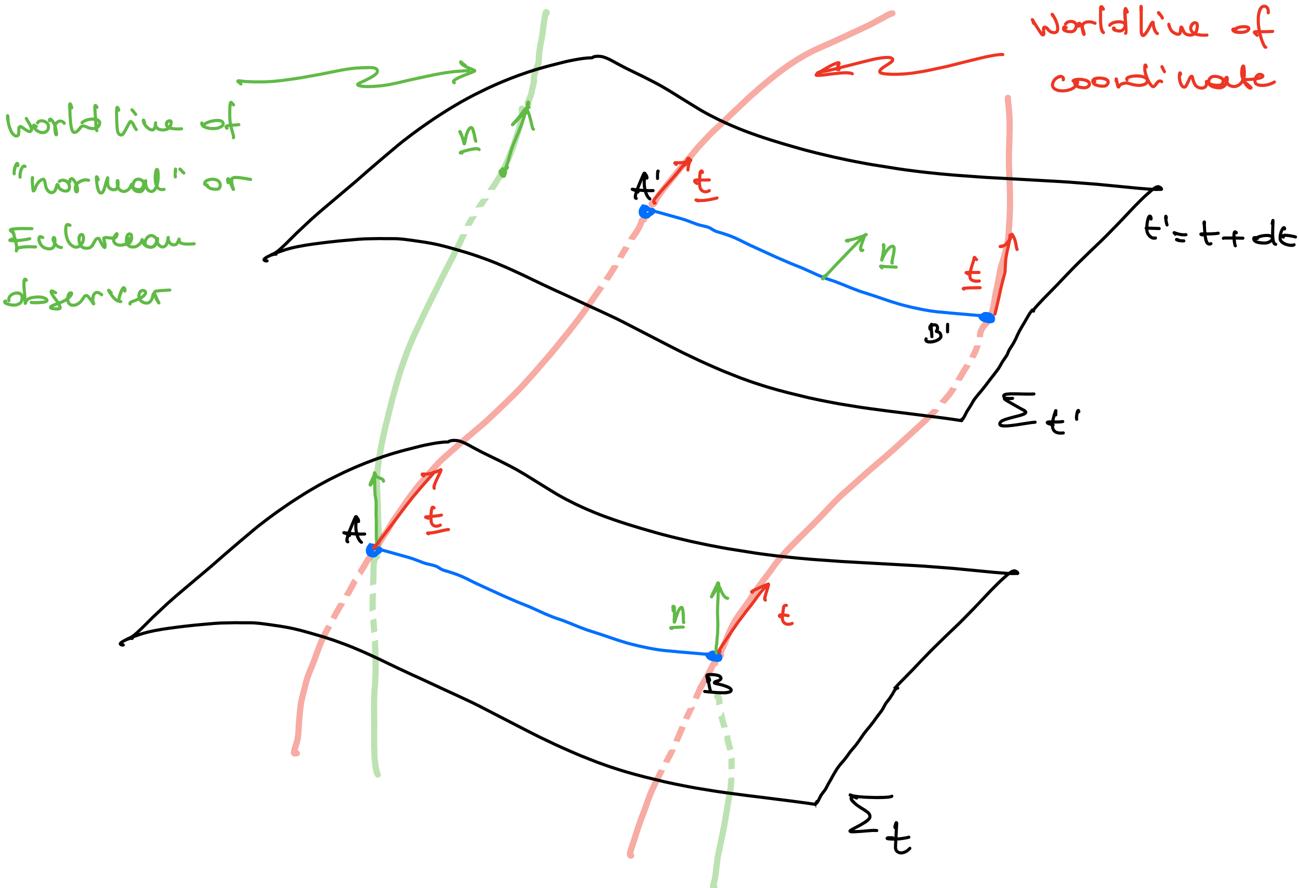
Let's interpret more carefully the lapse function.

Take the line element (25) and consider two events for which there is no change in spatial coordinates:

$$ds^2 = -\alpha^2 dt^2 = -d\tau^2 \implies$$

$d\tau = \pm \alpha \, dt$: the lapse function expresses the rate of change of proper time with respect to the coordinate time

Clearly: $d\tau = \pm dt$ for a flat spacetime.



An observer with tangent vector $\underline{n}$ is said to be a "normal" or "Eulerian" observer and is the standard observer in a 3+1 split and relative to this observer all quantities are computed. It's called Eulerian because it remains always at the same coordinate location.

This observer is also used to measure the velocity of a fluid in non-vacuum spacetimes. Let $\underline{u}$ be the 4-velocity, then

$$\vec{v} : \left(\begin{array}{c}\text{spatial} \\ \text{part of } \underline{u}\end{array}\right) = \frac{(\text{space})}{(\text{time})} := \frac{(\text{projection of } \underline{u} \text{ on } \Sigma_t)}{(\text{projection of } \underline{u} \text{ along } \underline{n})}$$

$$\boxed{v^i = \frac{\gamma^i{}_\mu u^\mu}{-u_\mu n^\mu} = \frac{\gamma^i{}_\mu u^\mu}{W}} \qquad (30)$$

where $W := -u_\mu n^\mu = \alpha u^t$: Lorentz factor

Indeed, it's possible to show that (exercise)

$$\boxed{W = (1 - v^i v_i)^{-1/2}}$$

so that $W \to \infty$ for $v^i \to 1$

It is possible to show that (exercise)

$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha} \qquad (31)$$

$$v_i = \frac{u_i}{W} = \gamma_{ij}\left(\frac{u^j}{W} + \frac{\beta^j}{\alpha}\right) \qquad (32)$$

Note how the shift just adds to the spatial part of u^i.

Let's compare (31) with the special relativistic equivalent :

$$v^i_{SR} := \frac{u^i}{u^t} = \frac{dx^i/dt}{dt/d\tau} = \frac{dx^i}{d\tau} = \frac{u^i}{W} \qquad (33) \qquad (SR)$$

vs

$\alpha = 1;\ \beta^i = 0$ in SR

$$v^i := \frac{u^i}{W} + \frac{\beta^i}{\alpha} = v^i_{SR} \qquad (GR)$$

clearly, the three-velocity gains a new term in general relativity that is related to the lapse and shift vectors, and also to the three-metric γ_{ij}

As a final remark we note that, as for any other tensor, also $\underline{u}$ can be decomposed in a purely time part and a purely space part:

$$u = W\left(\underline{n} + \underline{v}\right)$$

purely time (for $W\,\underline{n}$)

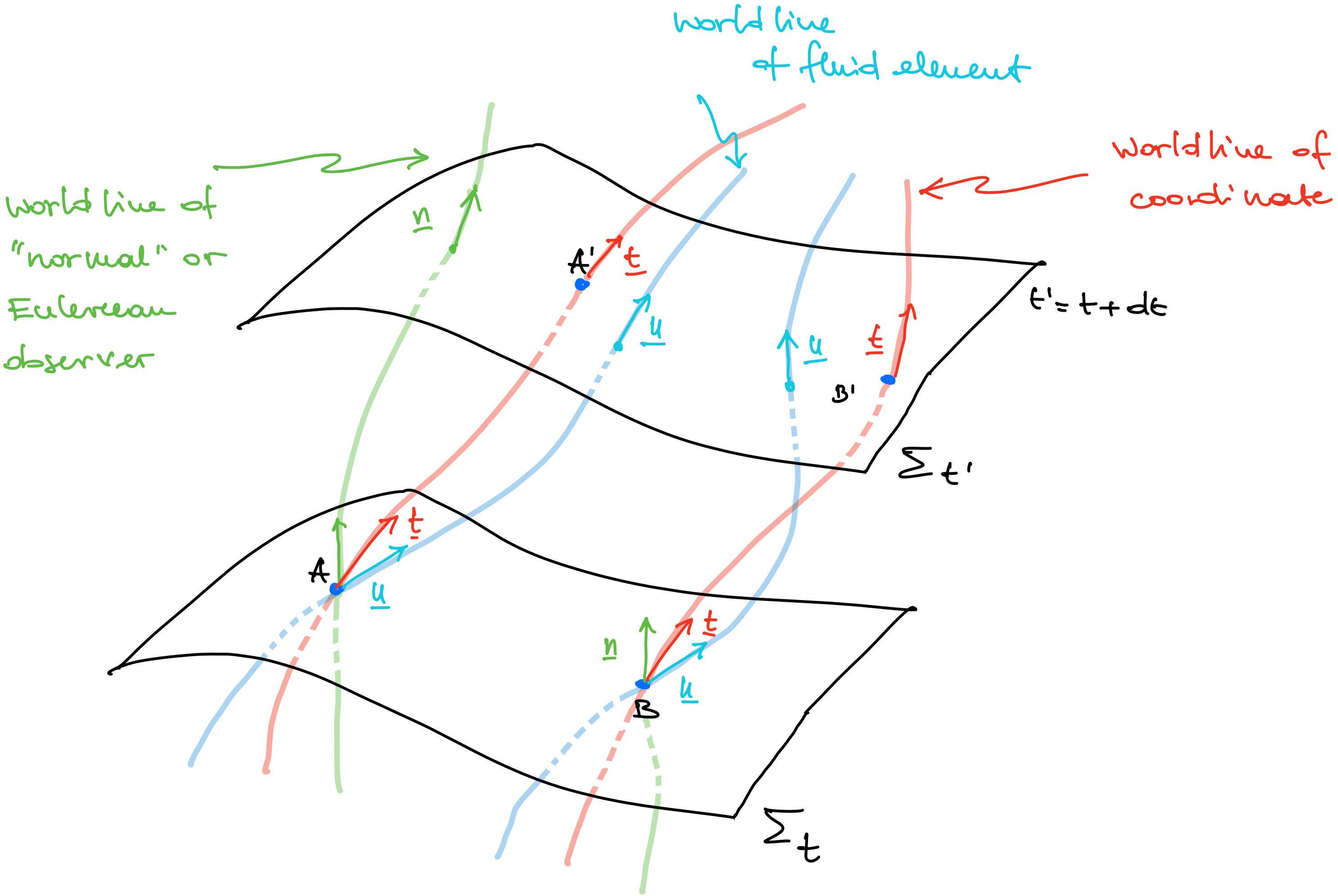
purely space part (for $W\,\underline{v}$)

As a concluding remark, I note that on every time slice Σ_t there will be three relevant timelike vectors:

$\underline{n}$: unit normal

$\underline{t}$: tangent to coordinate line

$\underline{u}$: " " fluid world line


world line of fluid element
world line of coordinate
world line of "normal" or Eulerean observer
n
A'
t
u
u
t
t' = t + dt
B'
$\Sigma_{t'}$
A
t
u
n
t
u
B
Σ_t