BIP, Timisoara May 2026

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Lecture Ia: 3+1 split in general relativity

$\boxed{\text{Lecture Ib: ADM formulation}}$

Lecture IIa: Relativistic Hydrodynamics

Lecture IIb: Numerical Methods for Hyperbolic PDEs

Lecture IIIa: Application to BNSs: bulk dynamics & GWs

Lecture IIIb: Application to BNSs: EM emission and neutrinos

Lecture Ib: ADM formulation

The ADM (Arnowitt, Deser, Misner 1962) was derived to obtain an Hamiltonian formulation of the Einstein equations, which have been first employed in numerical relativity. Much of my notation follows from the formulation of York 1979.

So far we have been dealing with tensor algebra in a 3+1 context. We need next to go to differential geometry; let ∇_μ be the covariant derivative in the 4D manifold; the spatial covariant derivative (or 3D covariant derivative) will be:

$$\boxed{D_\nu := \gamma^\mu{}_\nu \nabla_\mu = (\delta^\mu{}_\nu + n^\mu n_\nu)\nabla_\mu} \qquad (1)$$

Just like the 4D covariant derivative is compatible with the 4D metric, ie[⊙]

$$\boxed{\nabla_\mu g^{\mu\nu} = 0} \qquad (2)$$

⊙ $\nabla_\mu g^{\mu\nu} = 0$; take $\nabla_\mu u^\nu = \nabla_\mu (g^{\nu\alpha} u_\alpha) = (\nabla_\mu g^{\nu\alpha}) u_\alpha +$

$+ g^{\nu\alpha} \overbrace{\nabla_\mu u_\alpha}^{T_{\mu\alpha}} = (\nabla_\mu g^{\nu\alpha}) u_\alpha + g^{\nu\alpha} T_{\mu\alpha} = (\nabla_\mu g^{\nu\alpha}) u_\alpha + T_\mu{}^\nu$

$= (\nabla_\mu g^{\nu\alpha}) u_\alpha + \nabla_\mu u^\nu \Rightarrow \nabla_\mu g^{\nu\alpha} = 0$

So is the 3D covariant derivative compatible with the 3D metric, ie

$$\boxed{D_\mu \gamma^{\mu\nu} = 0} \qquad (3)$$

In practice, the 3D (spatial) covariant derivative of a tensor of rank k is the k+1 projection of the corresponding 4D covariant derivative. Eg

$$\boxed{D_\alpha T^\gamma{}_\beta = \gamma^\mu{}_\alpha \gamma^\rho{}_\beta \gamma^\gamma{}_\nu \nabla_\mu T^\nu{}_\rho} \, . \qquad (4)$$

What is important is that the spatial covariant derivative is built with 3D Christoffel symbols.

I recall that the Christoffel symbols are first-order derivatives of the metric,

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma} \right) \qquad (5)$$

So that the corresponding 3D (spatial) Christoffel symbols are

$$\boxed{{}^{(3)}\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\mu} \left(\partial_\beta \gamma_{\gamma\mu} + \partial_\gamma \gamma_{\mu\beta} - \partial_\mu \gamma_{\beta\gamma} \right)} \, . \qquad (6)$$

As for the 4D counterparts, also the 3D ones follow the same properties: they are symmetric on the lower indices and are not real tensors because they don't transform like tensors.

If we want to obtain a 3+1 formulation of the Einstein equations we have to follow the same mathematical route we have followed in 4D spacetimes

$$g_{\mu\nu} \to \Gamma^{\alpha}_{\mu\nu} \to \underset{\text{(Riemann)}}{R^{\alpha}{}_{\beta\mu\nu}} \to \underset{\text{(Ricci)}}{R_{\mu\nu}} \to \underset{\text{(Einstein)}}{G_{\mu\nu}} \tag{7}$$

$$\downarrow$$

$$\gamma_{\mu\nu} \to {}^{(3)}\Gamma^{\alpha}_{\mu\nu},\ \Gamma^{\alpha}_{\mu\nu} \to {}^{(3)}R^{\alpha}{}_{\beta\mu\nu} \to {}^{(3)}R_{\mu\nu} \to {}^{(3)}G_{\mu\nu} \tag{8}$$

Note:

$$ {}^{(3)}G_{\mu\nu} \neq \gamma^{\alpha}{}_{\mu}\,\gamma^{\beta}{}_{\nu}\,G_{\alpha\beta}$$

Since we need to proper define and employ differential operators acting on 3D objects on Σ_t.

So, let's start with the definition of the Riemann tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta}\Gamma^{\lambda}_{\nu\alpha} \tag{9}$$

$$= f\left(\partial^2 g, (\partial g)^2\right) ; \quad [R^{\mu}{}_{\nu\alpha\beta}] = L^{-2}$$

The corresponding spatial 3D curvature tensor will be

$$ {}^{(3)}R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha {}^{(3)}\Gamma^{\mu}{}_{\nu\beta} - \partial_\beta {}^{(3)}\Gamma^{\mu}{}_{\nu\alpha} + {}^{(3)}\Gamma^{\mu}{}_{\lambda\alpha}\, {}^{(3)}\Gamma^{\lambda}{}_{\nu\beta} - {}^{(3)}\Gamma^{\mu}{}_{\lambda\beta}\, {}^{(3)}\Gamma^{\lambda}{}_{\nu\alpha} \qquad (10) $$

where now ${}^{(3)}R^{\mu}{}_{\nu\alpha\beta}$ is purely spatial, ie

$$ \boxed{{}^{(3)}R^{\mu}{}_{\nu\alpha\beta}\, N^{\nu}{}_{\mu}\, N^{\alpha\beta} = 0} \qquad (11) $$

$N^{\alpha\beta} = N^{\alpha}{}_{\mu}\, \gamma^{\mu\beta}$

Similarly

$$ {}^{(3)}R^{\mu}{}_{\alpha\mu\beta} = {}^{(3)}R_{\alpha\beta} \quad : \text{3D spatial Ricci tensor} \qquad (12) $$

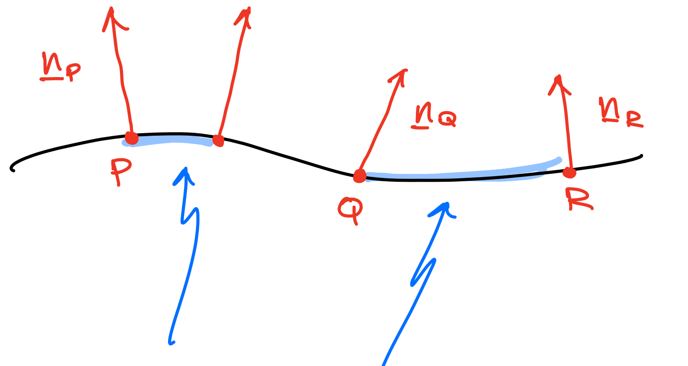
$$ {}^{(3)}R^{\alpha}{}_{\alpha} = {}^{(3)}R \quad : \text{3D spatial Ricci scalar} \qquad (13) $$

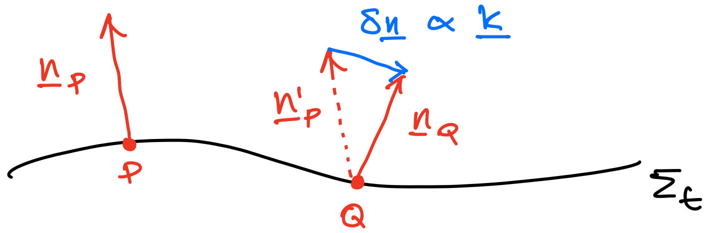
□

Note that when projecting the full 4D Rieman tensor over Σ_t and obtaining ${}^{(3)}R^{\mu}{}_{\nu\alpha\beta}$ we clearly lose some information, namely the information about how the hypersurface Σ_t is "curved" relative to the embedding 4D spacetime.

In other words, ${}^{(3)}R^{\mu}{}_{\nu\alpha\beta}$ tells us about the "intrinsic" curvature on Σ_t but does not inform us on the "extrinsic" curvature of Σ_t and we need to learn how to measure this curvature ${}^{(3)}\underline{K} = \underline{K}$.

It is geometrically very intuitive that the extrinsic curvature of a surface can be measured by comparing the normal vectors in two different positions:



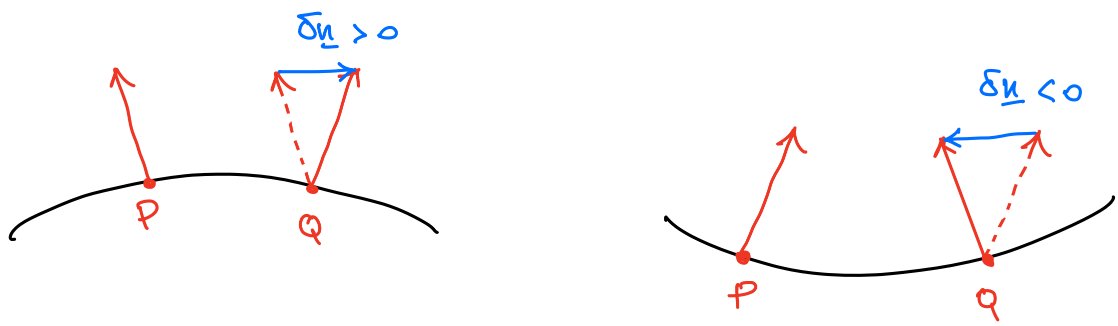


Let $\underline{n}_P$ be the normal vector at P and $\underline{n}'_P$ the same vector parallel transported at Q.

We can compare $\underline{n}'_P$ and $\underline{n}_Q$ and of course we are interested in the projection onto Σ_t of this difference. Let $\delta\underline{n}$ be the difference between $\underline{n}'_P$ and $\underline{n}_Q$, then the extrinsic curvature will be

$$\underline{k} := -\underline{\gamma} \cdot \delta\underline{n} \iff \boxed{k_{\mu\nu} = -\gamma^{\alpha}{}_{\mu} \nabla_\nu n_\alpha} \qquad (14)$$

As already anticipated, the extrinsic curvature can change sign depending on whethere the difference between the two normal vectors is positive (concave curvature) or negative (convex curvature).



As we will discuss later on, this is not the only possible definition of the extrinsic curvature.

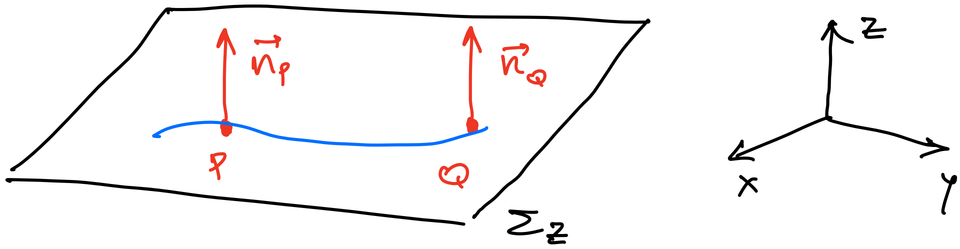
Intrinsic versus Extrinsic Curvature

As a useful way to grasp the differences between the intrinsic and extrinsic curvatures, let's use a simpler set up and replace $\mathcal{M}$ (4D manifold) with an Euclidean 3D spatial space (no time).

In this case, Σ_t is replaced by a 2D surface and we can consider different cases.

1. 2D plane in an Euclidean (flat) 3D space.

   Consider a Cartesian coordinate system (x, y, z) and let Σ_z be a $z =$ constant slice



The spatial metric γ_{ij} on Σ_z is diagonal and with components

$$\gamma_{ij} = \mathrm{diag}\,(1,1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ds^2 = \gamma_{ij}\, dx^i dx^j = dx^2 + dy^2 + dz^2$$

Obviously, the Riemann tensor is zero (in any coordinate system) and so is the intrinsic curvature; the normal three-vector $\vec{n}$ will have components

$$n^i = (0,0,1)\,; \quad n_i = (0,0,1)\,; \quad \underbrace{\vec{n}\cdot\vec{n} = 1}_{\text{unit vector}}$$

The extrinsic curvature will be

$$K_{ij} = -\gamma^k{}_j\, \nabla_i n_k \qquad i = 1,2$$

$$= -\gamma^3{}_j\, \underbrace{\nabla_i n_3}$$

$\partial_x n_3 = 0 = \partial_y n_3$

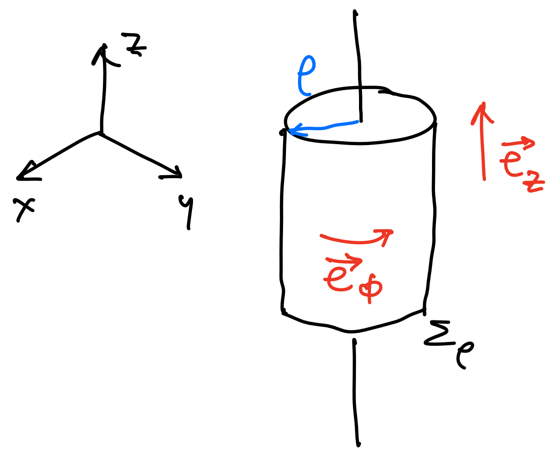
$$= 0$$

In other words the intrinsic and extrinsic curvature of Σ_z are zero. No surprise here...

Let's consider a more interesting 2-surface.

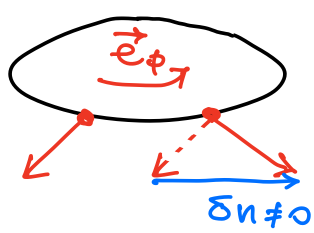
2. 2D cylinder in $\mathbb{R}^3$

Let's adopt a cylindrical coordinate system (ρ, ϕ, z) with $\rho = \bar{\rho} = \text{const.}$

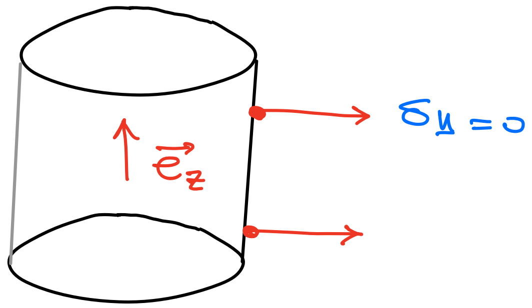


The intrinsic curvature on Σ_ρ is trivially also zero since we can "cut" Σ_ρ and it would lay on the plane without wrinkles.

The extrinsic curvature is more subtle and you can work it out that it is zero in one direction, i.e., the z-direction, but non-zero in the ϕ-direction.



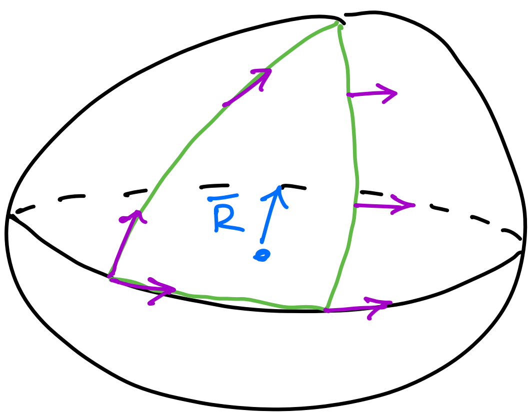
but non-zero in the ϕ-direction.



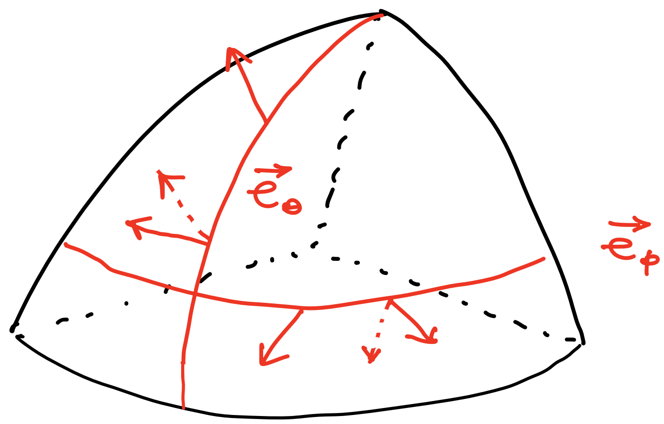
This example illustrates why the extrinsic curvature is not a scalar but a tensor: it can be different in different directions.

Finally, we can consider yet another example of a 2D surface in $\mathbb{R}^3$: a 2-sphere.

In this case, it's not difficult to show that the intrinsic curvature is non zero and indeed the Ricci Scalar $R = 1/\bar{R}^2 \neq 0$



At the same time, the extrinsic curvature is also nonzero in both the θ and ϕ-directions



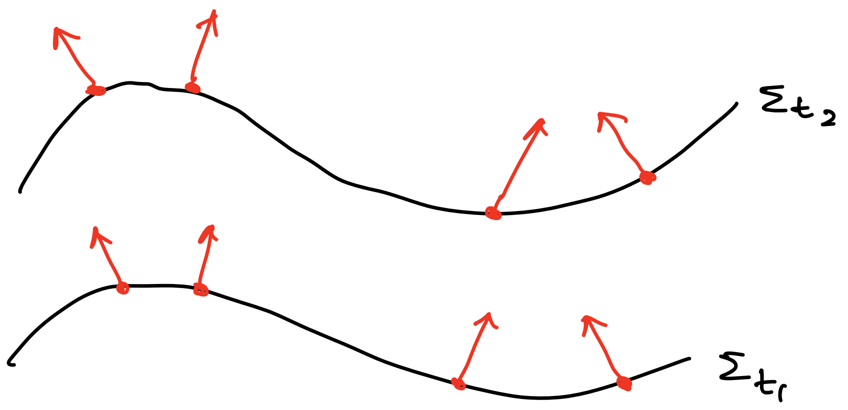
A second and equivalent manner to define the extrinsic curvature does not compare the unit normals to Σ_t at different locations, but rather looks at the same normal on different $t = $ const. slices. To this scope we need the acceleration of normal observers.

In full analogy with the acceleration of a fluid element (and hence of an observer comoving with the fluid) with unit four-velocity $\underline{u}$

$$\boxed{\tilde{a}_\mu := u^\nu \nabla_\nu u_\mu} \qquad (15)$$

we can define the acceleration of normal observers as

$$\boxed{a_\mu := n^\nu \nabla_\nu n_\mu} \qquad (16)$$



and subsequently define the extrinsic curvature as

$$\boxed{K_{\mu\nu} := -\nabla_\mu n_\nu - n_\mu a_\nu} \qquad (17)$$

The interpretation of this definition is also clear: Eulerian observers can be thought as "passive tracers" of spacetime curvature and the correspon-ding tangent vectors will converge or diverge depending on the extrinsic curvature.

Note that $\underline{K}$ is fully spatial :

$$\underline{K} \cdot \underline{n} = 0 = K^{\mu\nu} n_\mu \qquad (18)$$

There is a third and final definition of the extrinsic curvature and this involves the use of the Lie derivative. Let me refresh this concept.

The Lie derivative of a vector field $\underline{u}$ relative to another vector field $\underline{V}$ is given by

$$\mathcal{L}_{\underline{V}}\, \underline{u} := \nabla_{\underline{V}}\, \underline{u} - \nabla_{\underline{u}} V \tag{19}$$
$$= -[\underline{u}, \underline{V}] = [\underline{V}, \underline{u}]$$

In component form:

torsion-free spacetime

$$\left(\mathcal{L}_{\underline{V}}\, \underline{u}\right)^{\mu} = V^{\nu}\nabla_{\nu} u^{\mu} - u^{\nu}\nabla_{\nu} V^{\mu} = V^{\nu}\partial_{\nu} u^{\mu} - u^{\nu}\partial_{\nu} V^{\mu}$$

$$\left(\mathcal{L}_{\underline{V}}\, \underline{u}\right)_{\mu} = V^{\nu}\partial_{\nu} u_{\mu} + u_{\nu}\partial_{\mu} V^{\nu} .$$

Let me briefly recall the properties of the Lie derivative:

- $\mathcal{L}_{\phi \underline{V}}\, \underline{T} = \phi\, \mathcal{L}_{\underline{V}} T - \underline{V}\, \mathcal{L}_{\underline{T}}\, \phi$
- $\mathcal{L}_{\underline{V}}\, \phi = V^{\nu}\partial_{\nu}\phi$
- $\mathcal{L}_{\underline{V}}\left(a\underline{Y} + b\underline{Z}\right) = a\, \mathcal{L}_{\underline{V}}\, \underline{Y} + b\, \mathcal{L}_{\underline{V}}\, \underline{Z}$
- $\mathcal{L}_{\underline{V}}\left(\underline{Z}\, \underline{Y}\right) = \mathcal{L}_{\underline{V}}(\underline{Z})\, \underline{Y} + \underline{Z}\, \mathcal{L}_{\underline{V}}\, \underline{Y}$
- $\mathcal{L}_{\underline{V}}\left(T^{\alpha}{}_{\beta}\right) = V^{\mu}\partial_{\mu} T^{\alpha}{}_{\beta} - T_{\beta}{}^{\mu}\partial_{\mu} V^{\alpha} + T^{\alpha}{}_{\mu}\partial_{\beta} V^{\mu}$

We can now compute the Lie derivative of the spatial metric along the time normal; this is a sort of time derivative since $\underline{n}$ is where the time coordinate changes:

$$\mathcal{L}_{\underline{n}} \gamma_{\mu\nu} = n^\alpha \nabla_\alpha \gamma_{\mu\nu} + \gamma_{\mu\alpha} \nabla_\nu n^\alpha + \gamma_{\nu\alpha} \nabla_\mu n^\alpha$$

(see exercise)

$$= -2 K_{\mu\nu} \tag{20}$$

From this relation we conclude that

$$\boxed{K_{ij} = -\frac{1}{2} \mathcal{L}_{\underline{n}} \gamma_{ij}} \tag{21}$$

□

Recalling now that $\underline{t} = \alpha \underline{n} + \underline{\beta} \iff \alpha \underline{n} = \underline{t} - \underline{\beta}$

$\mathcal{L}_{\alpha \underline{n}} = \alpha \mathcal{L}_{\underline{n}}$

$$\mathcal{L}_{\underline{n}} = \frac{1}{\alpha} \mathcal{L}_{\alpha \underline{n}} = \frac{1}{\alpha} \left(\mathcal{L}_{\underline{t}} - \mathcal{L}_{\underline{\beta}} \right) = \frac{1}{\alpha} \left(\partial_t - \mathcal{L}_{\underline{\beta}} \right)$$

Stated differently, the Lie derivative along the timelike unit vector $\underline{n}$ is equivalent to the combined derivative along the time direction "plus" that along the generatrix of vector field $\underline{\beta}$.

As a result, we should expect evolution equations of the type

$$(\partial_t - \mathcal{L}_{\underline{\beta}})\, X = \cdots$$

where X can be a scalar, vector, tensor quantity living on Σ_t.

$$\mathcal{L}_{\underline{\beta}} \gamma_{\mu\nu} = \beta^\alpha \nabla_\alpha \gamma_{\mu\nu} + \gamma_{\mu\alpha} \nabla_\nu \beta^\alpha + \gamma_{\alpha\nu} \nabla_\mu \beta^\alpha$$
$$= \beta^\alpha \nabla_\alpha \cancel{n}_\mu n_\nu + \beta^\alpha \nabla_{\cancel{\alpha}} g_{\mu\nu} +$$
$$n_\mu n_{\cancel{\alpha}} \nabla_\nu \beta^\alpha + g_{\mu\alpha} \nabla_\nu \beta^\alpha +$$
$$n_\alpha n_{\cancel{\nu}} \nabla_\mu \beta^\alpha + g_{\alpha\nu} \nabla_\mu \beta^\alpha \overset{\beta \cdot \underline{n} = 0}{=}$$
$$= \nabla_\nu \beta_\mu + \nabla_\mu \beta_\nu \Rightarrow$$
$$\mathcal{L}_{\underline{\beta}} \gamma_{ij} = D_i \beta_j + D_j \beta_i$$

We can rewrite Eq. (21) as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\underline{\beta}} \gamma_{ij}$$
$$= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$
$$= \boxed{-2\alpha K_{ij} + 2 D_{(i} \beta_{j)} = \partial_t \gamma_{ij}} \qquad (22)$$

Note that Eq. (22) can be seen both as a derivative of K_{ij} and as a kinematical description of coordinates

(22) $\Longleftrightarrow$

$$\boxed{\left(\text{extrinsic curvature}\right) = \left(\begin{array}{l}\text{time derivative of} \\ \text{coordinates measured by} \\ \text{Eulerian observers}\end{array}\right)}$$

□

We will finally turn over to the Einstein equations and recall in what follows a number of identities derived well before Einstein's theory of General Relativity and that are generic equations of differential geometry resulting from the different possible combinations of projections onto Σ_t of the Riemann tensor.

We start with the Gauss-Codazzi equations, which are relative to a full spatial projection of the Riemann tensor $\underline{R}$

$\underline{\gamma} \cdot \underline{\gamma} \cdot \underline{\gamma} \cdot \underline{\gamma}\ \underline{R}$:

$$\boxed{\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\rho}_{\delta}\gamma^{\sigma}_{\lambda}R_{\mu\nu\rho\sigma} = {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta}K_{\beta\lambda} - K_{\alpha\lambda}K_{\beta\delta}} \tag{23}$$

Next, we consider the Codazzi-Mainardi equations that involve $\underline{\gamma} \cdot \underline{\gamma} \cdot \underline{\gamma} \cdot \underline{n} \cdot \underline{R}$

$$\boxed{\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\rho}_{\lambda}n^{\sigma}R_{\mu\nu\rho\sigma} = D_{\alpha}K_{\beta\lambda} - D_{\beta}K_{\alpha\lambda}} \tag{24}$$

and the Ricci equations that involve $\underline{\gamma} \cdot \underline{\gamma} \cdot \underline{n} \cdot \underline{n} \cdot \underline{R}$:

$$\boxed{\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}n^{\delta}n^{\lambda}R_{\alpha\delta\beta\lambda} = {}^{(3)}R_{\mu\nu} + K K_{\mu\nu} - K^{\lambda}{}_{\nu}K_{\mu\lambda}} \tag{25}$$

where

$$\boxed{K := K^{\mu}{}_{\mu}}$$

is the trace of the extrinsic curvature.

Combining the identities (23)-(25) we can write an evolution equation for the extrinsic curvature

$$\boxed{\begin{aligned}\partial_t K_{ij} = &-D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{ij}\partial_k \beta^k \\ &+ K_{il}\partial_j \beta^l \\ &+ \alpha\left({}^{(3)}R_{ij} + K K_{ij} - 2K_{il}K^l{}_j\right) \\ &+ \underbrace{4\pi\alpha\left[\gamma_{ij}(S-E) - 2S_{ij}\right]}\end{aligned}} \quad (26)$$

contributions coming from the energy-momentum tensor; these terms are zero in vacuum spacetimes

where

(27) $\boxed{S_{\mu\nu} := \gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu} T_{\alpha\beta}}$: spatial part of energy-momentum tensor

(28) $\boxed{S_{\mu} := -\gamma^{\alpha}{}_{\mu} n^{\beta} T_{\alpha\beta}}$: momentum density

(29) $\boxed{S := S^{\mu}{}_{\mu}}$: trace of $\mathcal{S}$

(30) $\boxed{E := n^{\alpha} n^{\beta} T_{\alpha\beta}}$: energy density measured by Eulerian observer

[$e := u^{\alpha}u^{\beta}T_{\alpha\beta}$: energy density in fluid comoving frame]

Note that the two evolution equations we have seen so far are for the 3-metric and the extrinsic curvature:

$$\begin{cases} \partial_t \gamma_{ij} = -2\alpha K_{ij} + \dots \\ \partial_t K_{ij} = -D_i D_j \alpha + \dots \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \partial_t x = v \\ \partial_t v = a = \partial_t^2 x \end{cases}$$

kinematics

dynamics

$\square$

Let's do some counting of the equations.

Einstein equations are 10 nonlinear second-order in time partial-differential equations [$G_{\mu\nu}$ is a rank-2 tensor (ie with 16 components) that is symmetric (ie only 10 are independent)].

What we have derived so far are:

$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \dots$: 6 equations [γ_{ij} has 9 components but 6 independent]

$\partial_t K_{ij} = -D_i D_j \alpha + \dots$: 6 equations

12 first-order in time eqs.

Because a second-order in time equation can be written as 2 first-order in time equations, we are short of 8 equations! This is because we have not considered all possible identities of the projections of the Riemann tensor and concentrated only on those involving a time derivative.

Let's consider the identity involving

$\underline{\gamma} \cdot \underline{\gamma} \cdot \underline{R} \longrightarrow$

$$\boxed{\gamma^{\alpha\mu} \gamma^{\beta\nu} R_{\alpha\beta\mu\nu} = 2 G_{\mu\nu} n^{\mu} n^{\nu}} \quad \Longrightarrow$$

$$\boxed{{}^{(3)}R + K^2 - K_{ij} K^{ij} = 16\pi E} \qquad (31)$$

(1 equation)

which is a generalisation of Gauss' "Theorema Egregium" relating intrinsic and extrinsic curvatures and

$\underline{\gamma} \cdot \underline{n} \cdot \underline{G} \longrightarrow$

$\cdots \gamma^{\alpha\mu} n^{\nu} g_{\mu\nu} R = (g^{\alpha\mu} + n^{\alpha} n^{\mu}) n^{\nu} g_{\mu\nu} R$
$= (n^{\alpha} - n^{\alpha}) R = 0$

$$\boxed{\gamma^{\alpha\mu} n^{\nu} G_{\mu\nu} = \gamma^{\alpha\mu} n^{\nu} R_{\mu\nu} = D^{\alpha} K - D_{\mu} K^{\alpha\mu}}$$

$\Longrightarrow$

$$\boxed{D_j \left(K^{ij} - \gamma^{ij} K\right) = 8\pi S^i} \qquad (32)$$

(3 equations)

Equations (31) and (32) do not have time derivatives and in fact they are elliptic equations. Because they express relations among the intrinsic and extrinsic curvatures at any time they are called "constrain equations".

In particular

(3) $$R + K^2 - K_{ij} K^{ij} = 16\pi E$$: Hamiltonian constraint

$$D_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^i$$: momentum constraint

As a result, the Einstein equations cast in a 3+1 form amount to 12 evolution equations ($\partial_t \gamma_{ij}$; $\partial_t K_{ij}$) and to 4 constraint equations.

A similar structure (evolution + constraints) is not new in physics. A very good example is given by the Maxwell equations, which I write in component and vector form:

$$\boxed{\begin{aligned} \partial_t E^i &= \epsilon^{ijk} \partial_j B^k - 4\pi J^i \\ \partial_t B^i &= -\epsilon^{ijk} \partial_j E^k \end{aligned}}$$

$$\boxed{\begin{aligned} \partial_i E^i &= 4\pi \rho_e \\ \partial_i B^i &= 0 \end{aligned}}$$

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B} - 4\pi \vec{J}$$

$$\partial_t \vec{B} = \vec{\nabla} \times \vec{E}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

The first six are clearly evolution equations for the electric and magnetic fields while the second six equations are constraints that must be satisfied at any time.

Just like for the Maxwell equations, where it is possible to show that if the constraints are satisfied initially, they are satisfied at all times, so for the Einstein equations it is possible to show that if the constraints are satisfied initially, the evolution preserves them. The problem is that this is true only when the constraints are satisfied exactly and this is never the case in numerical calculations.

The set of equations

$$\boxed{\begin{cases} \partial_t \gamma_{ij} = -2\alpha K_{ij} + \dots \\ \partial_t K_{ij} = -D_i D_j \alpha + \dots \end{cases}}$$

evolution equations

$$\boxed{\begin{cases} {}^{(3)}R + K^2 - K_{ij} K^{ij} = 16\pi E \\ D_j \left(K^{ij} - \gamma^{ij} K\right) = 8\pi S^i \end{cases}}$$

constraint equations

is also known as the ADM equations or the ADM formulation of the Einstein equations.

Essentially all numerical-relativity codes do not solve the constraint equations but at the initial time and to set the initial data. Only the evolution equations are actually solved and the violations of the constraint equations are only monitored (e.g., in terms of the L_2 norms).

For many years the ADM equations have been employed in numerical relativity for the solution of the Einstein equations. This has changed when experience and analytic calculations have shown they are weakly hyperbolic and not (strongly) hyperbolic.

As a result, they are not guaranteed to lead to well-posed initial-value problem. Indeed the numerical solution of the ADM equations rapidly leads to instabilities and blow-ups. We will see next how to fix the ADM equations and make them hyperbolic.

To make these statements less cryptic, let us recall that a system of first-order partial differential equations can always be cast in the form:

$$\boxed{\partial_t \underline{U} + \underline{\underline{A}}(\underline{U}) \nabla \underline{U} = \underline{S}} \qquad (1)$$

$\underline{U}$: state vector (vector of all evolved quantities)

$\underline{S}$: source vector

$\underline{\underline{A}}$: matrix of coefficients

- The system (1) is said to be linear if $\underline{\underline{A}}$ has constant coefficients and quasi-linear if $\underline{\underline{A}} = \underline{\underline{A}}(\underline{U})$.

- The system (1) is said to be (strongly) hyperbolic if $\underline{\underline{A}}$ is diagonalizable with a set of real eigenvalues and a set of linearly independent right eigenvectors:

$$\Lambda := R^{-1} A R \; ; \quad A R^{(i)} = \lambda_i R^{(i)} \; ; \quad \lambda_i \in \mathbb{R}$$

λ_i are also called "characteristics" or wave speeds and reflect that a hyperbolic system propagates waves.

- The system (1) is said to be weakly hyperbolic if $\underline{A}$ is not diagonalizable.

The importance of hyperbolicity is contained in a theorem that relates hyperbolicity to well-posedness. We recall that given:

- $\underline{u}(x,0)$: initial data of the system (1)
- $\underline{u}(x,t)$: solution of (1) at time $t>0$

then (1) is well-posed iff

$$\boxed{\|\underline{u}(x,t)\| \leq k e^{at} \|\underline{u}(x,0)\|} \qquad (2)$$

That is, the norm of $\underline{u}(x,t)$ (any norm) will grow in time at most exponentially. Of course one expects that $\|\underline{u}(x,t)\| \sim$ const but well-posedness guarantees that the solution will not "blow-up".

Clearly, what one wishes to have in a numerical solution.

With all the relevant definitions having been spilled out, we can finally enunciate the theorem:

$\boxed{\text{A (strongly) hyperbolic system is also well-posed.}}$