Relativistic Israel–Stewart Theory

Derivation from the 14-Moment Approximation

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Goal of the lecture

The goal is to derive the structure of relativistic Israel–Stewart theory from the relativistic Boltzmann equation using the 14-moment approximation.

The key transition is

$$
\begin{aligned}
\text{Navier–Stokes} \quad &: \quad \Pi, n^\mu, \pi^{\mu\nu} \text{ are algebraic in gradients,} \\
\text{Israel–Stewart} \quad &: \quad \Pi, n^\mu, \pi^{\mu\nu} \text{ are dynamical variables.}
\end{aligned}
$$

The final hydrodynamic theory consists of the conservation laws

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0,$$

plus relaxation-type equations for the dissipative currents.
The 14-moment method is different from the Chapmann-Enskog theory based on the Knudsen expansion. It is a finite-dimensional closure of the momentum dependence of the distribution function.

Historical background I: from Grad to relativistic moments

Grad introduced moment methods in nonrelativistic kinetic theory as an alternative to the Chapman–Enskog gradient expansion.

The central idea is to approximate the distribution function by finitely many momentum-space tensors:

$$1, \qquad k^{\mu}, \qquad k^{\mu}k^{\nu}, \qquad \dots$$

Instead of immediately eliminating non-equilibrium moments in favor of gradients, one keeps them as independent variables and derives their equations of motion.
This is why moment methods naturally lead to transient hydrodynamics: dissipative currents acquire their own evolution equations.

Historical background II: Israel and Stewart

Israel and Stewart adapted Grad's finite-moment idea to relativistic kinetic theory. Their construction is based on the observation that a dissipative relativistic fluid is specified by

$$N^\mu, \qquad T^{\mu\nu}.$$

Since N^μ has 4 components and $T^{\mu\nu}$ is symmetric with 10 components, the corresponding truncation contains

$$4 + 10 = 14$$

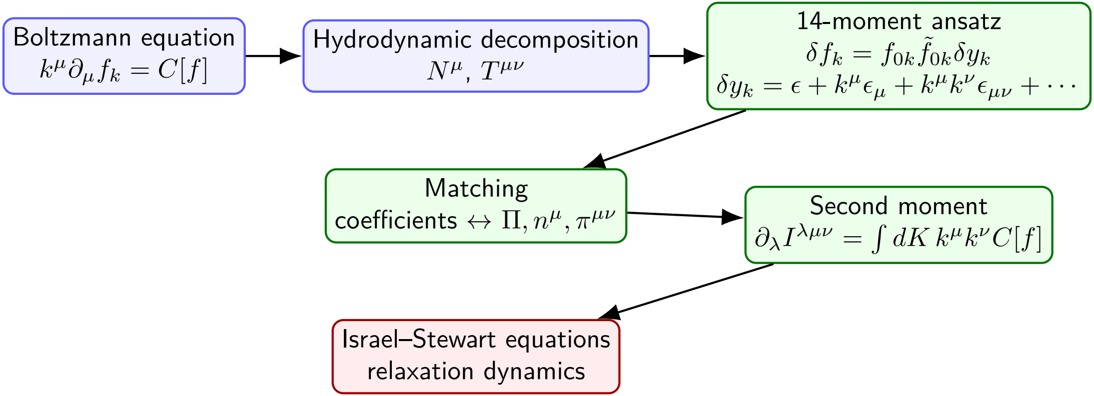
independent quantities.

The approximation is therefore designed to make

$$f_k = f_k(\alpha_0, \beta_0, u^\mu, \Pi, n^\mu, \pi^{\mu\nu}).$$

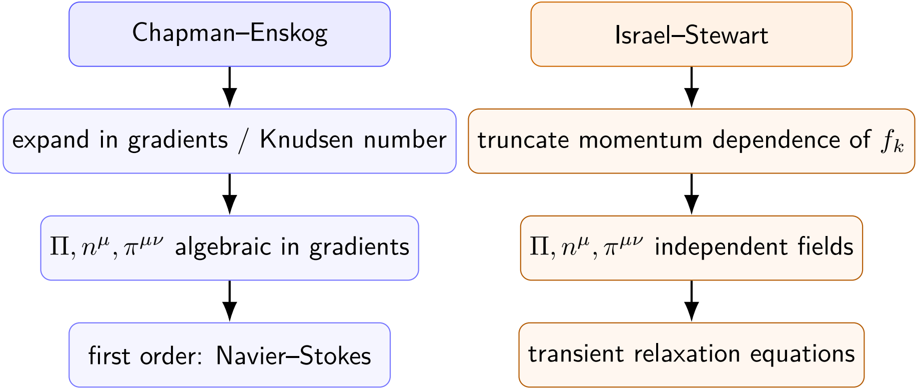
The number 14 is not arbitrary: it is due to the independent information contained in N^μ and $T^{\mu\nu}$.

Israel-Stewart theory from 14-moment approximation



Output: evolution equations for Π, n^μ, and $\pi^{\mu\nu}$.

Diagram: Chapman–Enskog vs Israel–Stewart



Conventions

The local-rest-frame energy is

$$E_k \equiv k^\mu u_\mu.$$

The spatial projector is

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

Irreducible projections are

$$A^{\langle\mu\rangle} = \Delta^\mu{}_\nu A^\nu,$$

$$A^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}.$$

All tensors are decomposed with respect to the local velocity u^μ. We work in the Landau frame.

Kinetic starting point

The relativistic Boltzmann equation is

$$k^\mu \partial_\mu f_k = C[f].$$

The conserved current and energy-momentum tensor are

$$N^\mu = \langle k^\mu \rangle \,, \qquad T^{\mu\nu} = \langle k^\mu k^\nu \rangle \,,$$

where

$$\langle \cdots \rangle = \int \mathrm{d}K \, (\cdots) f_k .$$

Collision invariants imply

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0.$$

These conservation laws are exact. Israel–Stewart theory adds evolution equations for non-conserved dissipative moments.

Hydrodynamic decomposition

Landau frame:

$$T^{\mu\nu} u_\nu = \varepsilon u^\mu .$$

The current and stress tensor are decomposed as

$$N^\mu = n u^\mu + n^\mu , \qquad u_\mu n^\mu = 0,$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu},$$

with

$$u_\mu \pi^{\mu\nu} = 0, \qquad \pi^\mu{}_\mu = 0.$$

The independent variables are α_0, β_0, the three independent components of u^μ, and the dissipative fields Π, n^μ, $\pi^{\mu\nu}$.

Hydrodynamic equations from conservation laws

Projecting $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ gives the equations for the primary hydrodynamic fields:

$$Dn + n\theta + \nabla_\mu n^\mu - n^\mu Du_\mu = 0,$$

$$D\varepsilon + (\varepsilon + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0,$$

$$(\varepsilon + P_0 + \Pi)Du^\alpha - \nabla^\alpha(P_0 + \Pi) - \pi^{\alpha\beta}Du_\beta + \Delta^\alpha_{\ \nu}\nabla_\mu\pi^{\mu\nu} = 0.$$

Here

$$\theta \equiv \partial_\mu u^\mu, \qquad \sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}.$$

These are not yet a closed system because Π, n^μ, and $\pi^{\mu\nu}$ still need equations of motion from Israel-Stewart theory.

Local equilibrium and matching

We have seen that the local-equilibrium distribution is

$$f_{0k} = \left[\exp(\beta_0 E_k - \alpha_0) + a\right]^{-1}, \qquad \beta_0 = 1/T, \qquad \alpha_0 = \mu/T.$$

Landau matching fixes

$$n = n_0 = \langle E_k \rangle_0 \,, \qquad \varepsilon = \varepsilon_0 = \left\langle E_k^2 \right\rangle_0 .$$

The pressure split is

$$P_0 = -\frac{1}{3} \left\langle \Delta^{\mu\nu} k_\mu k_\nu \right\rangle_0 , \qquad \Pi = -\frac{1}{3} \left\langle \Delta^{\mu\nu} k_\mu k_\nu \right\rangle_\delta .$$

Matching fixes the local thermodynamic state around which the non-equilibrium distribution is expanded.

Israel–Stewart distribution ansatz

The non-equilibrium single-particle distribution function is written as

$$f_k = \left[\exp(-y_k) + a\right]^{-1}.$$

The local-equilibrium value is

$$y_{0k} = \alpha_0 - \beta_0 u_\mu k^\mu = \alpha_0 - \beta_0 E_k.$$

The y_k is expanded in momentum space around its local equilibrium value

$$\delta y_k \equiv y_k - y_{0k} = \epsilon + k^\mu \epsilon_\mu + k^\mu k^\nu \epsilon_{\mu\nu} + k^\mu k^\nu k^\lambda \epsilon_{\mu\nu\lambda} + \cdots .$$

For small deviations,

$$f_k = f_{0k} + f_{0k}\tilde{f}_{0k}\delta y_k + \mathcal{O}(\delta y_k^2).$$

The expansion is in momentum tensors obtained by the 4-momentum k^μ (i.e. 1, k^μ, $k^\mu k^\nu, \cdots$), not in spacetime gradients.

The 14-moment truncation

The traditional approximation keeps only terms up to quadratic order:

$$\delta y_k \simeq \epsilon + k^\mu \epsilon_\mu + k^\mu k^\nu \epsilon_{\mu\nu}.$$

Without loss of generality, we have that is symmetric $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$ and traceless $\epsilon^\mu{}_\mu = 0$

Any antisymmetric part $\epsilon^A_{\mu\nu}$ of $\epsilon_{\mu\nu}$ does not contribute to $k^\mu k^\nu \epsilon_{\mu\nu}$ because $k^\mu k^\nu \epsilon^A_{\mu\nu} = 0$, and the trace can be absorbed into the scalar coefficient.

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Degree counting:

$$\epsilon : 1, \qquad \epsilon_\mu : 4, \qquad \epsilon_{\mu\nu} : 9, \qquad 1 + 4 + 9 = 14.$$

This is a finite polynomial approximation in momentum space. It retains precisely enough tensorial structures to represent scalar, vector, and rank-two dissipative distortions. The coefficients ε, ε_μ, and $\varepsilon_{\mu\nu}$ are not new hydrodynamic variables.
They can be related to the 14 components of N^μ and $T^{\mu\nu}$ using the matching procedure and, therefore, they will be related to the variable α_0, β_0, Π, u^μ, n^μ and $\pi^{\mu\nu}$.

Currents in the 14-moment approximation

Substituting $f_k = f_{0k} + f_{0k}\tilde{f}_{0k}(\epsilon + k^\mu \epsilon_\mu + k^\mu k^\nu \epsilon_{\mu\nu})$ into the definitions of N^μ and $T^{\mu\nu}$ gives

$$\begin{aligned} N^\mu &= I_0^\mu + \epsilon J_0^\mu + J_0^{\mu\nu}\epsilon_\nu + J_0^{\mu\nu\lambda}\epsilon_{\nu\lambda}, \\ T^{\mu\nu} &= I_0^{\mu\nu} + \epsilon J_0^{\mu\nu} + J_0^{\mu\nu\lambda}\epsilon_\lambda + J_0^{\mu\nu\lambda\rho}\epsilon_{\lambda\rho}. \end{aligned}$$

Notice that the ideal parts are given by $N^\mu_{\rm ideal} = I_0^\mu$ and $T^{\mu\nu}_{\rm ideal} = I_0^{\mu\nu}$
Obviously the other terms related to $J_0^{\alpha_1,\cdots\alpha_n}$ comes from the dissipative corrections to f_k

Where we have introduced the tensors

$$\begin{aligned} I_0^{\alpha_1\cdots\alpha_n} &= \int dK\, k^{\alpha_1}\cdots k^{\alpha_n} f_{0k}, \\ J_0^{\alpha_1\cdots\alpha_n} &= \int dK\, k^{\alpha_1}\cdots k^{\alpha_n} f_{0k}\tilde{f}_{0k}. \end{aligned}$$

The tensor structure depends only on u^μ and $g^{\mu\nu}$

$$\begin{aligned} I_0^\mu &= I_{10}u^\mu, \qquad J_0^\mu = J_{10}u^\mu, \\ I_0^{\mu\nu} &= I_{20}u^\mu u^\nu - I_{21}\Delta^{\mu\nu}, \\ J_0^{\mu\nu} &= J_{20}u^\mu u^\nu - J_{21}\Delta^{\mu\nu}. \end{aligned}$$

Matching equations

The 14 coefficients (ϵ,ϵ^{μ}, $\epsilon^{\mu\nu}$) are fixed by

$$\Delta^{\mu}{}_{\nu} N^{\nu} = n^{\mu},$$

$$\Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} = \pi^{\mu\nu},$$

$$-\frac{1}{3}\Delta_{\mu\nu}(T^{\mu\nu} - T^{\mu\nu}_{\mathrm{ideal}}) = \Pi,$$

$$u_{\mu}(N^{\mu} - N^{\mu}_{\mathrm{ideal}}) = 0, \qquad u_{\nu}(T^{\mu\nu} - T^{\mu\nu}_{\mathrm{ideal}}) = 0.$$

The first three equations define the dissipative currents. The last two impose particle matching and the Landau-frame condition.

Quantity	Constraint	Independent components
α_0, β_0, Π	scalar quantities	3
u^{μ}	$u_{\mu} u^{\mu} = 1$	3
n^{μ}	$u_{\mu} n^{\mu} = 0$	3
$\pi^{\mu\nu}$	symmetric, traceless, transverse ($u_{\mu}\pi^{\mu\nu} = 0$)	5

Dissipative currents in terms of coefficients ϵ,ϵ^{μ},$\epsilon^{\mu\nu}$

The relation between dissipative fields and kinetic coefficients are given:

$$
\begin{aligned}
\Pi &= J_{21}\epsilon + J_{31}u^{\lambda}\epsilon_{\lambda} + \left(J_{41} + \frac{5}{3}J_{42}\right)u^{\lambda}u^{\rho}\epsilon_{\lambda\rho},\\
n^{\mu} &= -J_{21}\Delta^{\mu\nu}\epsilon_{\nu} - 2J_{31}\Delta^{\mu\nu}u^{\lambda}\epsilon_{\nu\lambda},\\
\pi^{\mu\nu} &= 2J_{42}\Delta^{\mu\nu}_{\lambda\rho}\epsilon^{\lambda\rho}.
\end{aligned}
$$

Exercise: Prove the above equations.
This set of equations can be inverted as

$$
\begin{aligned}
\epsilon &= E_0\Pi,\\
\epsilon_{\lambda} &= D_0\Pi u_{\lambda} + D_1 n_{\lambda},\\
\epsilon_{\lambda\rho} &= B_0(\Delta_{\lambda\rho} - 3u_{\lambda}u_{\rho})\Pi + B_1 u_{(\lambda}n_{\rho)} + B_2\pi_{\lambda\rho}.
\end{aligned}
$$

Solution of the matching system

Defining

$$D_{nq} = J_{n+1,q}J_{n-1,q} - J_{nq}^2, \qquad J_{nq} \equiv \frac{(-1)^q}{(2q+1)!!}\left\langle \tilde{f}_{0k}(k^\mu u_\mu)^{n-2q}(\Delta^{\mu\nu}k_\mu k_\nu)^q \right\rangle_0$$

The vector and tensor sectors give

$$B_1 = \frac{J_{31}}{D_{31}}, \qquad D_1 = -\frac{J_{41}}{D_{31}}, \qquad B_2 = \frac{1}{2J_{42}}.$$

For the scalar sector,

$$\frac{E_0}{3B_0} = m^2 + 4\frac{J_{31}J_{30} - J_{41}J_{20}}{D_{20}} \equiv -C_1,$$

$$\frac{D_0}{3B_0} = -4\frac{J_{31}J_{20} - J_{41}J_{10}}{D_{20}} \equiv -C_2,$$

$$B_0 = -\frac{1}{3C_1J_{21} + 3C_2J_{31} + 3J_{41} + 5J_{42}}.$$

After this step, the distribution function is closed in terms of the 14 hydrodynamic variables.

Closed 14-moment distribution

The distribution function becomes

$$f_k = f_{0k} + f_{0k}\tilde{f}_{0k}\delta y_k,$$

with

$$\delta y_k = E_0\Pi + k^\lambda(D_0\Pi u_\lambda + D_1 n_\lambda) + k^\lambda k^\rho \epsilon_{\lambda\rho},$$

$$\epsilon_{\lambda\rho} = B_0(\Delta_{\lambda\rho} - 3u_\lambda u_\rho)\Pi + B_1 u_{(\lambda}n_{\rho)} + B_2\pi_{\lambda\rho}.$$

Any non-equilibrium moment computed with this distribution is now expressible through Π, n^μ, and $\pi^{\mu\nu}$.

The correction can be written schematically as

$$\delta f_k = f_{0k}\tilde{f}_{0k}\left[\mathcal{A}_\Pi(E_k)\Pi + \mathcal{A}_n(E_k)k_{\langle\mu\rangle}n^\mu + \mathcal{A}_\pi(E_k)k_{\langle\mu}k_{\nu\rangle}\pi^{\mu\nu}\right].$$

- Bulk pressure Π: scalar distortion.
- Particle diffusion n^μ: vector distortion.
- Shear stress $\pi^{\mu\nu}$: traceless tensor distortion.

The distribution is no longer exactly isotropic in the local rest frame. The 14-moment approximation parameterizes the leading non-equilibrium distortions by macroscopic dissipative currents.

Equations of motion for dissipative currents

After the 14-moment approximation, the non-equilibrium correction δf_k is expressed in terms of the dissipative currents

$$\Pi, \qquad n^\mu, \qquad \pi^{\mu\nu}.$$

The remaining task is to determine their equations of motion. In the traditional Israel–Stewart derivation, this is done by taking the second moment of the Boltzmann equation:

$$\partial_\lambda I^{\lambda\mu\nu} = \int dK\, k^\mu k^\nu C[f], \qquad I^{\lambda\mu\nu} = \int dK\, k^\lambda k^\mu k^\nu f_k.$$

The 14-moment ansatz fixes the form of δf_k, but it does not uniquely determine which moment of the Boltzmann equation should be used to obtain the evolution equations for the dissipative currents.

The use of the second moment is a prescription of the traditional Israel–Stewart construction. Following the paper by Denicol (2010)[a], one may instead derive the equations of motion directly from the kinetic definitions of Π, n^μ, and $\pi^{\mu\nu}$.

[a] G. S. Denicol, T. Koide, and D. H. Rischke, Phys. Rev. Lett. **105**, 162501 (2010).

Kinetic equations for dissipative currents

From the definitions of the dissipative currents,

$$D\Pi = -\frac{1}{3}m^2 \int \mathrm{d}K \, D\delta f_k,$$

$$Dn^{\langle\mu\rangle} = \int \mathrm{d}K \, k^{\langle\mu\rangle} D\delta f_k,$$

$$D\pi^{\langle\mu\nu\rangle} = \int \mathrm{d}K \, k^{\langle\mu} k^{\nu\rangle} D\delta f_k.$$

Using the Boltzmann equation,

$$D\delta f_k = -Df_{0k} - \frac{1}{E_k} k^{\langle\mu\rangle} \nabla_\mu f_k + \frac{1}{E_k} C[f].$$

These exact equations become hydrodynamic equations once the 14-moment approximation is inserted into all non-equilibrium moments.

$$D\Pi + C = -\frac{m^2}{3}\alpha_0^s\theta - \left(\frac{2}{3} - \frac{m^2}{3}\frac{G_{20}}{D_{20}}\right)\Pi\theta - \frac{m^2}{3}\frac{G_{20}}{D_{20}}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{m^2}{3}\frac{G_{30}}{D_{20}}\partial_\mu n^\mu$$
$$+ \frac{m^4}{9}\left\langle E_k^{-2}\right\rangle_\delta \theta + \frac{m^2}{3}\left\langle E_k^{-2}k^{\langle\mu}k^{\nu\rangle}\right\rangle_\delta \sigma_{\mu\nu} + \frac{m^2}{3}\nabla_\mu\left\langle E_k^{-1}k^{\langle\mu\rangle}\right\rangle_\delta .$$

$$Dn^{\langle\mu\rangle} - C^\mu = \alpha_0^v\nabla^\mu\alpha_0 + n_\nu\omega^{\mu\nu} - n^\mu\theta - \frac{3}{5}n_\nu\sigma^{\mu\nu} + \frac{\beta_0 J_{21}}{\epsilon_0 + P_0}\left(\Pi Du^\mu - \nabla^\mu\Pi - \pi^{\mu\nu}Du_\nu + \Delta^\mu{}_\nu\nabla_\lambda\pi^{\lambda\nu}\right)$$
$$- \frac{m^2}{3}\left\langle E_k^{-2}k^{\langle\mu\rangle}\right\rangle_\delta \theta - \Delta^\mu{}_\lambda\nabla_\nu\left\langle E_k^{-1}k^{\langle\lambda}k^{\nu\rangle}\right\rangle_\delta - \frac{2m^2}{5}\left\langle E_k^{-2}k^{\langle\nu\rangle}\right\rangle_\delta \sigma^\mu{}_\nu$$
$$- \frac{m^2}{3}\nabla^\mu\left\langle E_k^{-1}\right\rangle_\delta - \left\langle E_k^{-2}k^{\langle\mu}k^\nu k^{\lambda\rangle}\right\rangle_\delta \sigma_{\lambda\nu}.$$

$$D\pi^{\langle\mu\nu\rangle} - C^{\mu\nu} = \alpha_0^t\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda + \frac{6}{5}\Pi\sigma^{\mu\nu}$$
$$- \frac{4m^2}{7}\Delta^{\mu\nu}_{\alpha\beta}\left\langle E_k^{-2}k^{\langle\lambda}k^{\alpha\rangle}\right\rangle_\delta \sigma^\beta{}_\lambda - \frac{2m^4}{15}\left\langle E_k^{-2}\right\rangle_\delta \sigma^{\mu\nu} - \frac{2m^2}{5}\Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha\left\langle E_k^{-1}k^{\langle\beta\rangle}\right\rangle_\delta$$
$$- \frac{m^2}{3}\left\langle E_k^{-2}k^{\langle\mu}k^{\nu\rangle}\right\rangle_\delta \theta - \left\langle E_k^{-2}k^{\langle\mu}k^\nu k^\lambda k^{\rho\rangle}\right\rangle_\delta \sigma_{\lambda\rho} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\left\langle E_k^{-1}k^{\langle\alpha}k^\beta k^{\lambda\rangle}\right\rangle_\delta .$$

These equations are exact but not yet closed. The 14-moment approximation is then used to express those moments, and the collision integrals C, C^μ, $C^{\mu\nu}$, in terms of Π, n^μ, and $\pi^{\mu\nu}$.

Collision projections and relaxation terms

Define

$$C = \frac{1}{3}m^2 \int \mathrm{d}K\, E_k^{-1} C[f],$$

$$C^\mu = \int \mathrm{d}K\, E_k^{-1} k^{\langle\mu\rangle} C[f],$$

$$C^{\mu\nu} = \int \mathrm{d}K\, E_k^{-1} k^{\langle\mu} k^{\nu\rangle} C[f].$$

Within the 14-moment approximation,

$$C = \frac{4}{3} m^2 B_0 A_{02}^s \Pi,$$

$$C^\mu = -B_1 \frac{A_{01}^v}{\beta_0} n^\mu,$$

$$C^{\mu\nu} = -B_2 \frac{A_{00}^t}{\beta_0^2} \pi^{\mu\nu}.$$

These terms generate the relaxation times of the dissipative currents.

Linearized collision integrals

keeping only the first-order non-equilibrium correction to the distribution function the collision integrals entering the equations of motion are evaluated with the *linearized collision operator*.

Up to terms of order $\mathcal{O}(\delta y_k^2)$, one obtains

$$C = \frac{m^2}{3\nu} \int dK\, dK'\, dP\, dP'\, \frac{1}{E_k} W_{kk'\to pp'} f_{0k} f_{0k'} \tilde{f}_{0p} \tilde{f}_{0p'} \left(y_p + y_{p'} - y_k - y_{k'}\right), \tag{3.152}$$

$$C^{\mu} = \frac{1}{\nu} \int dK\, dK'\, dP\, dP'\, \frac{k^{\langle\mu\rangle}}{E_k} W_{kk'\to pp'} f_{0k} f_{0k'} \tilde{f}_{0p} \tilde{f}_{0p'} \left(y_p + y_{p'} - y_k - y_{k'}\right), \tag{3.153}$$

$$C^{\mu\nu} = \frac{1}{\nu} \int dK\, dK'\, dP\, dP'\, \frac{k^{\langle\mu}k^{\nu\rangle}}{E_k} W_{kk'\to pp'} f_{0k} f_{0k'} \tilde{f}_{0p} \tilde{f}_{0p'} \left(y_p + y_{p'} - y_k - y_{k'}\right). \tag{3.154}$$

The equilibrium contribution drops out because $y_{0k} = \alpha_0 - \beta_0 E_k$ is built from collision invariants. Hence

$$y_{0p} + y_{0p'} - y_{0k} - y_{0k'} = 0,$$

and only the first-order deviation from equilibrium contributes to C, C^{μ}, and $C^{\mu\nu}$.

Israel–Stewart equation for bulk pressure

The bulk-pressure equation has the form

$$\tau_\Pi D\Pi = -\Pi - \zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}$$

$$-\ell_{\Pi n}\nabla_\mu n^\mu - \tau_{\Pi n} n^\mu D u_\mu - \lambda_{\Pi n} n^\mu \nabla_\mu \alpha_0.$$

Equivalently,

$$\tau_\Pi D\Pi + \Pi = -\zeta\theta + \text{second-order couplings.}$$

The first two terms describe relaxation toward the Navier–Stokes value $\Pi_{\rm NS} = -\zeta\theta$.

Israel–Stewart equation for diffusion

The diffusion-current equation has the form

$$\tau_n D n^{\langle\mu\rangle} = -n^\mu + \kappa \nabla^\mu \alpha_0 - \tau_{nn} n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \lambda_{nn} n_\nu \sigma^{\mu\nu}$$

$$+\ell_{n\pi} \Delta^{\mu\nu} \nabla_\lambda \pi^\lambda{}_\nu - \tau_{n\pi} \pi^\mu{}_\nu D u^\nu - \lambda_{n\pi} \pi^{\mu\nu} \nabla_\nu \alpha_0$$

$$-\ell_{n\Pi} \nabla^\mu \Pi + \tau_{n\Pi} \Pi D u^\mu + \lambda_{n\Pi} \Pi \nabla^\mu \alpha_0.$$

Neglecting relaxation and second-order terms gives the Navier–Stokes limit $n^\mu \simeq \kappa \nabla^\mu \alpha_0$.

Israel–Stewart equation for shear stress

The shear-stress equation has the form

$$\tau_\pi D\pi^{\langle\mu\nu\rangle} = -\pi^{\mu\nu} + 2\eta\sigma^{\mu\nu} + 2\tau_{\pi\pi}\pi^{\langle\mu}{}_{\alpha}\omega^{\nu\rangle\alpha}$$
$$-\delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\langle\mu}{}_{\alpha}\sigma^{\nu\rangle\alpha} - \tau_{\pi n}n^{\langle\mu}Du^{\nu\rangle}$$
$$+\ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}.$$

The leading relaxation structure is

$$\tau_\pi D\pi^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots .$$

Complete hydrodynamic system

The Israel–Stewart system is the conservation sector

$$Dn + n\theta + \nabla_\mu n^\mu - n^\mu Du_\mu = 0,$$

$$D\varepsilon + (\varepsilon + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0,$$

$$(\varepsilon + P_0 + \Pi)Du^\alpha - \nabla^\alpha(P_0 + \Pi) - \pi^{\alpha\beta}Du_\beta + \Delta^\alpha_{\ \nu}\nabla_\mu\pi^{\mu\nu} = 0,$$

including the transient sector

$$\tau_\Pi D\Pi + \Pi = -\zeta\theta + \cdots,$$

$$\tau_n Dn^{\langle\mu\rangle} + n^\mu = \kappa\nabla^\mu\alpha_0 + \cdots,$$

$$\tau_\pi D\pi^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots.$$

This is now a closed system and a causal theory once an equation of state and all transport coefficients are supplied.

First-order transport coefficients

The first-order transport coefficients obtained in this approach are given

$$\boxed{\zeta = \frac{\alpha_0^s}{4B_0 A_{02}^s}}$$

$$\boxed{\kappa = \frac{\beta_0 \alpha_0^v}{B_1 A_{01}^v}}$$

$$\boxed{\eta = \frac{\beta_0^2 \alpha_0^t}{2B_2 A_{00}^t}}.$$

These coefficients are controlled by the same scalar, vector, and tensor collision matrices that appear in Chapman–Enskog theory.

Relaxation times

The relaxation times are

$$\boxed{\tau_\Pi = \frac{3}{4m^2 B_0 A^s_{02}}}$$

$$\boxed{\tau_n = \frac{\beta_0}{B_1 A^v_{01}}}$$

$$\boxed{\tau_\pi = \frac{\beta_0^2}{B_2 A^t_{00}}}.$$

The corresponding ratios are

$$\frac{\tau_\Pi}{\zeta} = \frac{3}{m^2 \alpha^s_0}, \qquad \frac{\tau_n}{\kappa} = \frac{1}{\alpha^v_0}, \qquad \frac{\tau_\pi}{\eta} = \frac{2}{\alpha^t_0}.$$

Finite relaxation times are the main new physical ingredient compared with first-order relativistic Navier–Stokes theory. Notice that the ratios between the relaxation times and the corresponding transport coefficients are thermodynamical quantities which are independent of the collision term.

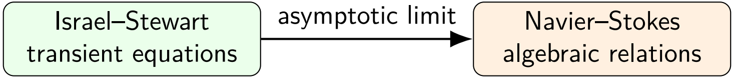
Navier–Stokes limit inside Israel–Stewart

At late times, or when microscopic relaxation is fast compared with macroscopic evolution,

$$\tau_i \dot{X}_i \ll X_i.$$

Then the Israel–Stewart equations reduce to

$$\Pi \simeq -\zeta\theta, \qquad n^\mu \simeq \kappa \nabla^\mu \alpha, \qquad \pi^{\mu\nu} \simeq 2\eta\sigma^{\mu\nu}.$$



Israel–Stewart theory contains Navier–Stokes as a limiting regime, but does not impose instantaneous constitutive relations. Dissipative currents relax toward the Navier–Stokes values over microscopic time scales.

Physical interpretation

The 14-moment approximation turns the Boltzmann equation into a finite-dimensional dynamical system for

$$\alpha_0, \beta_0, u^\mu, \Pi, n^\mu, \pi^{\mu\nu}.$$

The conservation laws evolve the primary fields, while the relaxation equations evolve the dissipative fields.

This is the kinetic origin of transient relativistic dissipative hydrodynamics.
The approximation is not exact: it truncates the momentum dependence of δy_k. Modern formulations improve this by using systematic moment expansions and power counting.

Backup slides

References

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Linear system for matching coefficients

Substitution gives

$$J_{21}D_1 + J_{31}B_1 = -1, \qquad 2J_{42}B_2 = 1,$$

$$J_{21}E_0 + J_{31}D_0 - (3J_{41} + 5J_{42})B_0 = 1,$$

$$J_{10}E_0 + J_{20}D_0 - 3(J_{30} + J_{31})B_0 = 0,$$

$$J_{31}D_1 + J_{41}B_1 = 0,$$

$$J_{20}E_0 + J_{30}D_0 - 3(J_{40} + J_{41})B_0 = 0.$$

Technical comment. The system separates into scalar, vector, and tensor sectors, reflecting the decomposition of Π, n^{μ}, and $\pi^{\mu\nu}$.