BIP Relativistic Fluid Dynamics
Kinetic Theory - Lecture 1 Notes

Salvatore Plumari

Timişoara, 25-29 May 2026

Relativistic Boltzmann Equation

Single non-degenerate gas

We will introduce the basic concepts of relativistic kinetic theory and the relativistic Boltzmann equation that describes the time evolution of the phase-space distribution function. We start by considering a single non-degenerate relativistic gas, i.e. a gas where quantum effects are not taken into account.
Consider a gas of on-shell particles with rest mass m.

We define spacetime coordinates and momentum four-vector:

$$x = (t, \mathbf{x}), \qquad k^\mu = (k^0, \mathbf{k}) \tag{1}$$

with:

$$k^0 = \sqrt{\mathbf{k}^2 + m^2} \tag{2}$$

We call $f(x, k) = f(\mathbf{x}, \mathbf{k}, t)$ the phase-space distribution function:

$$f(\mathbf{x}, \mathbf{k}, t)\, d^3\mathbf{x}\, d^3\mathbf{k} = f(\mathbf{x}, \mathbf{k}, t)\, dx^1 dx^2 dx^3 dk^1 dk^2 dk^3, \tag{3}$$

which represents the number of particles in the volume element $d^3\mathbf{x}$ around $\mathbf{x}$, with momenta in the range $d^3\mathbf{k}$ around $\mathbf{k}$ at time t.
The number of particles in a volume element is a *scalar invariant*, since all observers will count the same number of particles in any reference frame. The phase-space volume $d\mu = d^3\mathbf{x}\, d^3\mathbf{k}$ is a Lorentz invariant as well (see Exercise 0). This means that the phase-space distribution function $f(x, k)$ is a Lorentz invariant, due to the fact that:

$$dN = f(x, k, t)\, d^3x\, d^3\mathbf{k}. \tag{4}$$

Change in particle number

Now we want to calculate the total change in the particle number per unit time interval $\dfrac{\Delta N}{\Delta t}$

Let $d\mu(t) = d^3\mathbf{x}(t)\, d^3\mathbf{k}(t)$ be the volume element at time t; the number of particles in this volume is:

$$N(t) = f(\mathbf{x}, \mathbf{k})\, d\mu(t) \tag{5}$$

At time $t+\Delta t$ the volume becomes $d\mu(t+\Delta t) = d^3\mathbf{x}(t+\Delta t)\, d^3\mathbf{k}(t+\Delta t)$ and the number of particles:

$$N(t+\Delta t) = f(\mathbf{x}+\Delta\mathbf{x}, \mathbf{k}+\Delta\mathbf{k}, t+\Delta t)\, d\mu(t+\Delta t) \tag{6}$$

The collisions between particles imply that $N(t) \neq N(t+\Delta t)$ and the change in particle number is:

$$\Delta N = N(t+\Delta t) - N(t) = f(\mathbf{x}+\Delta\mathbf{x}, \mathbf{k}+\Delta\mathbf{k}, t+\Delta t)\, d\mu(t+\Delta t) - f(\mathbf{x}, \mathbf{k})\, d\mu(t) \tag{7}$$

The variations of the coordinates are given by:

$$\Delta\mathbf{x} = \mathbf{v}\Delta t = \frac{\mathbf{k}}{k^0}\Delta t, \qquad \Delta\mathbf{k} = \mathbf{F}\Delta t \tag{8}$$

where
$\mathbf{F}(\mathbf{x}, \mathbf{k}, t)$ is the external force
The transformed volume element is:

$$d\mu(t+\Delta t) = |J|\, d\mu(t) \tag{9}$$

where J is the Jacobian of the transformation.
We compute the Jacobian:

$$J = \det \begin{pmatrix} \frac{\partial x^1(t+\Delta t)}{\partial x^1(t)} & \frac{\partial x^2(t+\Delta t)}{\partial x^1(t)} & \cdots & \frac{\partial k^3(t+\Delta t)}{\partial x^1(t)} \\ \frac{\partial x^1(t+\Delta t)}{\partial x^2(t)} & \frac{\partial x^2(t+\Delta t)}{\partial x^2(t)} & \cdots & \frac{\partial k^3(t+\Delta t)}{\partial x^2(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^1(t+\Delta t)}{\partial k^3(t)} & \frac{\partial x^2(t+\Delta t)}{\partial k^3(t)} & \cdots & \frac{\partial k^3(t+\Delta t)}{\partial k^3(t)} \end{pmatrix} \tag{10}$$

Using:

$$x^i(t+\Delta t) = x^i(t) + v^i\Delta t = x^i(t) + \frac{k^i}{k^0}\Delta t \tag{11}$$

$$k^i(t+\Delta t) = k^i(t) + F^i\Delta t \tag{12}$$

The partial derivatives appear as:

$$\frac{\partial x^i(t+\Delta t)}{\partial x^j} = \delta^{ij}, \tag{13}$$

$$\frac{\partial x^i(t+\Delta t)}{\partial k^j} = \frac{\partial\left(k^i/k^0\right)}{\partial k^j}\Delta t, \tag{14}$$

$$\frac{\partial k^i(t+\Delta t)}{\partial k^j(t)} = \delta^{ij} + \frac{\partial F^i}{\partial k^j}\Delta t. \tag{15}$$

As we can see, if we take only linear terms the Jacobian can be written as:

$$J = \det(\mathbb{1} + \hat{A}\,\Delta t) = 1 + \Delta t\,\mathrm{Tr}\hat{A} + \mathcal{O}(\Delta t^2) = \left(1 + \Delta t \sum_i \frac{\partial F^i}{\partial k^i}\right) + \mathcal{O}(\Delta t^2) \tag{16}$$

Therefore:

$$d\mu(t+\Delta t) = \left(1 + \Delta t \frac{\partial F^i}{\partial k^i}\right) d\mu(t). \tag{17}$$

Let us now expand $f(\mathbf{x}+\Delta\mathbf{x}, k+\Delta\mathbf{k}, t+\Delta t)$ in Taylor series around the point $(\mathbf{x}, \mathbf{k}, t)$ and keeping only linear terms in Δt:

$$f(\mathbf{x}+\Delta\mathbf{x}, \mathbf{k}+\Delta\mathbf{k}, t+\Delta t) = f(\mathbf{x}, \mathbf{k}, t) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x^i}\Delta x^i + \frac{\partial f}{\partial k^i}\Delta k^i + \mathcal{O}(\Delta t^2) \tag{18}$$

Using:

$$\Delta x^i = \dot{x}^i \Delta t, \qquad \Delta k^i = F^i \Delta t \tag{19}$$

we get:

$$f(\mathbf{x}+\Delta\mathbf{x}, \mathbf{k}+\Delta\mathbf{k}, t+\Delta t) = f(\mathbf{x}, \mathbf{k}, t) + \Delta t\left[\frac{\partial f}{\partial t} + \dot{x}^i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial k^i}\right]. \tag{20}$$

The change of the number of particles is therefore

$$\Delta N = f(\mathbf{x}+\Delta\mathbf{x}, \mathbf{k}+\Delta\mathbf{k}, t+\Delta t)\, d\mu(t+\Delta t) - f(\mathbf{x}, \mathbf{k}, t)\, d\mu(t) = \tag{21}$$

$$= \left[f(\mathbf{x}, \mathbf{k}, t) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x^i} v^i \Delta t + \frac{\partial f}{\partial k^i} F^i \Delta t\right] d\mu(t+\Delta t) - f(\mathbf{x}, \mathbf{k}, t) d\mu(t). \tag{22}$$

Now, combining with the change of volume element:

$$d\mu(t+\Delta t) = \left(1 + \Delta t \frac{\partial F^i}{\partial k^i}\right) d\mu(t), \tag{23}$$

we obtain:

$$\frac{\Delta N}{\Delta t} = \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial k^i} + f \frac{\partial F^i}{\partial k^i}\right] d\mu(t) = \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \frac{\partial (f F^i)}{\partial k^i}\right] d\mu(t) \tag{24}$$

We know that ΔN is a scalar invariant and the proper time satisfies:

$$\Delta \tau = \frac{\Delta t}{\gamma} \tag{25}$$

Therefore:

$$\frac{\Delta N}{\Delta \tau} = \gamma \frac{\Delta N}{\Delta t} = \gamma \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \frac{\partial (f F^i)}{\partial k^i}\right] d\mu(t) \tag{26}$$

is a scalar invariant quantity.

Let us rewrite the previous result in covariant form. The first two terms can be written by multiplying and dividing by k^0 as:

$$\frac{\gamma}{k^0}\left[k^0 \frac{\partial f}{\partial t} + k^0 \frac{k^i}{k^0} \frac{\partial f}{\partial x^i}\right] = \frac{\gamma}{k^0} k^\alpha \frac{\partial f}{\partial x^\alpha} = \frac{k^\alpha}{m} \frac{\partial f}{\partial x^\alpha}, \tag{27}$$

where we exploited $k^0/\gamma = k^0_{\mathrm{LRF}} = m$.

To rewrite the force term, we introduce the Minkowski force:

$$\mathcal{K}^\alpha = \frac{dk^\alpha}{d\tau} \tag{28}$$

and one can show that:

$$0 = \mathcal{K}^\alpha k_\alpha = \mathcal{K}^0 k_0 - \mathcal{K}^i k_i \tag{29}$$

One can prove that

$$\gamma \frac{\partial (f F^i)}{\partial k^i} = \frac{\partial (f \mathcal{K}^\alpha)}{\partial k^\alpha} \tag{30}$$

Finally, we obtain:

$$\frac{\Delta N}{\Delta t} = \frac{1}{k^0} \left[k^\alpha \frac{\partial f}{\partial x^\alpha} + m \frac{\partial (f \mathcal{K}^\alpha)}{\partial k^\alpha} \right] d\mu(t) \tag{31}$$

1 Collision term

Now we have to find an expression for $\frac{\Delta N}{\Delta t}$.

We decompose it into two contributions:

$$\frac{\Delta N}{\Delta t} = \left(\frac{\Delta N}{\Delta t} \right)_{\text{gain}} - \left(\frac{\Delta N}{\Delta t} \right)_{\text{loss}} \tag{32}$$

These two terms describe two physical mechanisms:

- Gain: particles entering the phase-space element $d\mu = d^3\mathbf{x}\, d^3\mathbf{k}$ in Δt
- Loss: particles leaving the phase-space element $d\mu = d^3\mathbf{x}\, d^3\mathbf{k}$ in Δt

To calculate the collision integral we have to do some assumptions:

- Only binary collisions take place

  $$k + k' \to p + p', \tag{33}$$

  which is a good approximation if the gas is dilute.

- Molecular chaos (*Stosszahlansatz*):

  $$f^{(2)}(x, k, k') = f(x, k)\, f(x, k'), \tag{34}$$

  where we assume to be able to factorise the *two*-particle distribution function $f^{(2)}(x, k, k')$ as the product of two one-particle distribution functions. It means that the momenta $\mathbf{k}$ and $\mathbf{k}'$ of two particles before the collision are considered uncorrelated, and the same is true for the momenta $\mathbf{p}$ and $\mathbf{p}'$ afterwards.

- Slow variation of the one particle distribution function $f(x, k)$ over a time interval which is larger than the duration of a collision but smaller than the time between collisions:

  $$\tau_{\text{collision}} \ll \tau_{\text{macro}} \tag{35}$$

- Locality of collisions: only particles in the same position $\mathbf{x}$ can collide.

Loss term

The loss term accounts for all collisions in which a particle with incoming momentum $\mathbf{k}$ collides with another particle of momentum $\mathbf{k}'$ and is scattered into some different outgoing momentum $\mathbf{p}$, while the other one is scattered to a momentum $\mathbf{p}'$.

$$\mathbf{k} + \mathbf{k}' \to \mathbf{p} + \mathbf{p}' \tag{36}$$

Notice that after the collision the particle is no longer in the momentum state $\mathbf{k}$, so the contribution given to the evolution of $f(x, k)$ is negative.

Under the molecular chaos assumption, the density of incoming pairs is simply given by the product of the one-particle distribution functions $f(x, k)\, f(x, k')$. Therefore the number of pairs that will collide is

$$f(x, k)d^3\mathbf{k}\, f(x, k')d^3\mathbf{k}'. \tag{37}$$

Let us now call

$$W(k, k' \mid p, p') \tag{38}$$

the transition rate from the incoming pair (k, k') to the outgoing pair (p, p'). The total probability that at time t a pair of particles leaves the state (k, k') to the state (p, p') is:

$$\mathbb{P}_{kk' \to pp'} = W(k, k' \mid pp')\, dP\, dP' \Delta t \tag{39}$$

Therefore, the number of collisions removing particles from the state k is:

$$dN_{\text{loss}} = f(x, k)\, f(x, k')\, W(k, k' \mid p, p')\, dK\, dK'\, dP\, dP'\, \Delta t \tag{40}$$

Integrating over all possible final momenta:

$$\left(\frac{\Delta N}{\Delta t\, dK}\right)_{\text{loss}} = \int dK'\, dP\, dP'\, f(x, k)\, f(x, k')\, W(k, k' \mid p, p') \tag{41}$$

To obtain the influence of the collisions on the distribution function, we consider the number of particles in the range $\Delta^4 x\, \Delta^3 k$, which changes as the results of the collision by an amount which can be written in a Lorentz covariant way as:

$$\Delta^4 x \frac{\Delta^3 \mathbf{k}}{k^0} C(x, k)$$

and hence $C(x, k)$ is an invariant.

To obtain a properly relativistic expression integrating also on 4-momentum, we must introduce the invariant integration measure. Since we consider on-shell particles, the integration over 4-momentum is constrained by the condition $k^2 = m^2$ and $k^0 > 0$. We can write:

$$d^4k\, \delta(k^2 - m^2)\, \theta(k^0) = \frac{d^4k}{2k^0} \delta(k^0 - \sqrt{\mathbf{k}^2 + m^2}) \tag{42}$$

Thus, the invariant phase-space measure becomes:

$$\int d^4k\, \theta(k^0)\, \delta(k^2 - m^2) = \frac{1}{2} \int \frac{d^3\mathbf{k}}{k^0} \tag{43}$$

Therefore, the loss term can be written as:

$$\left(\frac{dN}{dt}\right)_{\text{loss}} = \frac{1}{2}\Delta^3 x \frac{\Delta^3 k}{k^0} \int \frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}\, f(x,k)\, f(x,k')\, W(k,k' \mid p,p') \tag{44}$$

Here we include the factor 1/2 to take into account the fact that a final state with momenta (p, p') cannot be distinguished from another with momenta (p', p).

Gain term

We now consider the inverse process:

$$p + p' \to k + k' \tag{45}$$

This corresponds to particles entering the state k.

The number of incoming pairs, now with momenta (p, p'), is:

$$f(x,p)\, f(x,p') dP\, dP' \tag{46}$$

The transition probability is given by:

$$\mathbb{P}_{pp' \to kk'} = W(p,p' \mid k,k')\, dK\, dK'\, dt \tag{47}$$

Therefore, the number of particles gained in the state k after the collision is:

$$dN_{\text{gain}} = f(x,p)\, f(x,p')\, W(p,p' \mid k,k')\, dK\, dK'\, dP\, dP'\, dt \tag{48}$$

Integrating over all incoming states:

$$\left(\frac{dN}{dt\, dK}\right)_{\text{gain}} = \int dK'\, dP\, dP'\, f(x,p)\, f(x,p')\, W(p,p' \mid k,k') \tag{49}$$

The gain of particles in the range $\Delta^4 x$ and $\Delta^3 k$ around $\mathbf{x}$ and $\mathbf{k}$ with initial momentum $(\mathbf{p}, \mathbf{p}') \to (\mathbf{k}, \mathbf{k}')$ is:

$$\frac{1}{2}\Delta^4 x \frac{\Delta^3 \mathbf{k}}{k^0} \int \frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0} f(x,p)\, f(x,p')\, W(p,p' \mid k,k') \tag{50}$$

Combining gain and loss terms, we obtain:

$$\begin{aligned} C(x,k) &= \left(\frac{dN}{dt}\right)_{\text{gain}} - \left(\frac{dN}{dt}\right)_{\text{loss}} = \\ &= \frac{1}{2}\int \frac{d^3\mathbf{k}'}{k'^0}\,\frac{d^3\mathbf{p}}{p^0}\,\frac{d^3\mathbf{p}'}{p'^0}\Big[f(x,p)f(x,p')\, W(p,p' \mid k,k') - f(x,k)f(x,k')\, W(k,k' \mid p,p')\Big] \end{aligned} \tag{51}$$

Eventually, we get the Boltzmann Equation:

$$k^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial (f\mathcal{K}^\alpha)}{\partial k^\alpha} = \frac{1}{2}\int \frac{d^3\mathbf{k}'}{k'^0}\,\frac{d^3\mathbf{p}}{p^0}\,\frac{d^3\mathbf{p}'}{p'^0}\Big[f_p f_{p'}\, W(p,p' \mid k,k') - f_k f_{k'}\, W(k,k' \mid p,p')\Big] \tag{52}$$

1.1 Properties of the transition rate $W(k, k'|p, p')$

For notational convenience, let us write

$$W(k, k'|p, p') \equiv W_{kk'|pp'}, \tag{53}$$

where $W(k, k'|p, p')$ denotes the Lorentz-invariant transition rate for the binary elastic collision $(k, k') \longrightarrow (p, p')$ The transition rate has two important symmetry properties: i) the labels of the two incoming particles are arbitrary. In fact, exchanging $k \leftrightarrow k'$ does not define a different physical initial state and therefore a different process, but only a different labelling of the same two-particle state. b) the labels of the two outgoing particle are arbitrary. For similar reasoning. exchanging $p \leftrightarrow p'$ does not change the physical final state. Therefore,

$$W_{k,k'|p,p'} = W_{k',k|p,p'} = W_{k,k'|p',p} = W_{k',k|p',p}. \tag{54}$$

Furthermore, microscopic reversibility implies that the direct and inverse collisions have the same transition rate:

$$W_{k,k'|p,p'} = W_{p,p'|k,k'} \tag{55}$$

This property follows from time-reversal invariance of the microscopic collision dynamics. It means that the probability for the process $(k, k') \to (p, p')$ is the same as the probability for the inverse process $(p, p') \to (k, k')$. These symmetry properties of W are important in order to get the conservation laws, as we will see later.

In general, the transition rate must satisfy energy-momentum conservation:

$$W(k, k' \mid p, p') = s\, \sigma(s, \theta)\, \delta^{(4)}(k + k' - p - p') \tag{56}$$

where:

- $\sigma(s, \theta)$ is the cross section and has the dimension of an area
- $s = (k + k')^2$ is the Mandelstam variable

The differential cross section is defined such that its product with the flux of incoming particles in the initial state gives the transition probability per unit volume and time.
In a Lorentz frame the flux of particles is given by the relative velocity

$$v_{rel} = \frac{\mathcal{F}}{k\, k'}, \quad \text{with } \mathcal{F} = \sqrt{(k \cdot k')^2 - m^4} = \frac{1}{2}\sqrt{s(s - 4m^2)}, \tag{57}$$

$\mathcal{F}$ is called invariant flux and is manifestly Lorentz-invariant, as visible by the RHS.

The transition rate can be written as:

$$W(k, k' \mid p, p') \frac{d^3\mathbf{p}}{p^0} \frac{d^3\mathbf{p}'}{p'^0}. \tag{58}$$

Therefore:

$$d\sigma = W(k, k' \mid p, p') \frac{d^3\mathbf{p}}{p^0} \frac{d^3\mathbf{p}'}{p'^0} \frac{1}{F} \tag{59}$$

Integrating both sides over final state momenta and using Eq.(56) one finds:

$$\int d\sigma = \int \frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}\,\frac{1}{F} s\,\sigma(s,\theta)\,\delta^4(k+k'-p-p') \tag{60}$$

In the centre-of-mass frame:

$$\mathbf{p}'+\mathbf{p}=0, \qquad p_0=p_0', \qquad p^0=\sqrt{s}/2 \tag{61}$$

and thus it is easy to perform one of the two integrations:

$$\int d\sigma = 2\int d^3\mathbf{p}\frac{1}{\mathcal{F}}\delta\left(\frac{\sqrt{s}}{2}-p^0\right)\sigma(s,\theta) \tag{62}$$

The integration measure is $d^3\mathbf{p}=|\mathbf{p}|^2\,d|\mathbf{p}|\,d\Omega$ which leads to the following shape for the δ:

$$\delta\left(\frac{\sqrt{s}}{2}-\sqrt{|\mathbf{p}|^2+m^2}\right)=\sqrt{\frac{s}{s-4m^2}}\delta\left(|\mathbf{p}|-\frac{1}{2}\sqrt{s-4m^2}\right) \tag{63}$$

This allows to perform also the integration over $|\mathbf{p}|$:

$$\int d\sigma = \int d\Omega\,\sigma(s,\theta) \tag{64}$$

Therefore, the collision integral can be expressed in terms of the differential cross section:

$$C[f]=\frac{1}{2}\int\frac{d^3\mathbf{k}'}{k'^0}\,d\Omega\,\mathcal{F}\left[f_pf_{p'}-f_kf_{k'}\right]\sigma(s,\theta) \tag{65}$$

where $\mathcal{F}$ is the invariant flux and $\sigma(s,\theta)$ is the differential cross section.

Thus, the collision term is fully determined by the microscopic cross section and we can write the Relativistic Boltzmann Equation as:

$$k^\alpha\frac{\partial f}{\partial x^\alpha}+m\frac{\partial f\mathcal{K}^\alpha}{\partial k^\alpha}=\frac{1}{2}\int\frac{d^3\mathbf{k}'}{k'^0}d\Omega\,\left[f_pf_{p'}-f_kf_{k'}\right]\,\sigma(s,\theta) \tag{66}$$

Conservation Laws

The macroscopic quantities that describe a relativistic system out of equilibrium obey equations that follow from conservation laws of particle number and energy-momentum.

In a purely macroscopic theory these are postulated, while in kinetic theory they can be derived.

1.2 Lemma for a single-component gas

We consider a single-component gas.

The transport equation reads:

$$k^\mu\partial_\mu f(x,k)=C(x,k) \tag{67}$$

It is possible to prove that the collision term satisfies the following fundamental property:

$$F[\psi]:=\int\frac{d^3\mathbf{k}}{k^0}\,\psi(x,k)\,C(x,k)=0 \tag{68}$$

where $\psi(x,k) \equiv \psi_k$ is any function of the form:

$$\psi(x,k) = a(x) + b_\mu(x)\, k^\mu \tag{69}$$

with $a(x)$ scalar functions and $b_\mu(x)$ a four-vector, both of which are constant with respect to k and $a_k(x)$ satisfy:

$$a_k(x) + a_{k'}(x) = a_p(x) + a_{p'}(x). \tag{70}$$

Here for example for a_k we mean that the subscript does not refer to a momentum dependence, but means that the quantity a_k is referred to the particle with momentum k.

Substituting the explicit form of the collision term Eq.(52) in Eq. (68):

$$F[\psi] = \frac{1}{2}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}\psi_k \left[f_p f_{p'}\, W_{pp'|kk'} - f_k f_{k'}\, W_{kk'|pp'}\right] \tag{71}$$

if we take the second addend and exchange $k \leftrightarrow p, k' \leftrightarrow p'$, we get:

$$\psi_k f_k f_{k'} W_{kk'|pp'} \leftrightarrow \psi_p f_p f_{p'} W_{pp'|kk'}.$$

By exploiting the symmetry property of the transition probability $W_{pp'|kk'}$:

$$F[\psi] = \frac{1}{2}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}(\psi_k - \psi_p) f_p f_{p'}\, W_{pp'|kk'} \tag{72}$$

If we now exchange $p \leftrightarrow p', k \leftrightarrow k'$ we get:

$$F[\psi] = \frac{1}{2}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}(\psi_{k'} - \psi_{p'}) f_p f_{p'}\, W_{p'p|k'k} \tag{73}$$

By exploiting $W_{p'p|k'k} = W_{pp'|kk'}$, one can say:

$$F[\psi] = \frac{(72)+(73)}{2} = \frac{1}{4}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}(\psi_k + \psi_{k'} - \psi_p - \psi_{p'}) f_p f_{p'}\, W_{pp'|kk'} \tag{74}$$

Using now:

- Conservation of energy-momentum:

$$p_k^\mu + p_l^\mu = p_k'^\mu + p_l'^\mu \tag{75}$$

- Constraint on the scalar functions:

$$a_k + a_l = a_k' + a_l'; \tag{76}$$

we obtain:

$$\psi_k + \psi_{k'} - \psi_p - \psi_{p'} = 0 \tag{77}$$

Therefore:

$$\int \frac{d^3\mathbf{k}}{k^0}\,\psi_k\, C_k = 0 \tag{78}$$

which is the proof of the Lemma.

The quantities ψ_k that fulfil this Lemma are called summational invariants. Quite interestingly, the Lemma can be proved also in the opposite direction: every summational invariant quantity $\psi(x,k)$ can be written as: $\psi(x,k) = a + b^\mu k_\mu$.

2 General equation of transfer for a single-component gas

Let us consider the Boltzmann equation for a single-component gas in the case with no external forces $K^\alpha = 0$:

$$k^\mu \partial_\mu f(x,k) = C(x,k) \tag{79}$$

We multiply both sides by an arbitrary function $\psi(x,k)$ and integrate over momentum:

$$\int \frac{d^3\mathbf{k}}{k^0}\, \psi(x,k)\, k^\mu \partial_\mu f(x,k) = \int \frac{d^3\mathbf{k}}{k^0}\, \psi(x,k)\, C(x,k) \tag{80}$$

Let us now work on the two terms separately.
We explicit the expression of the collision integral in the RHS by using the result previously found in Eq. (74)

$$\text{RHS} = \frac{1}{4}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}(\psi_k + \psi_{k'} - \psi_p - \psi_{p'}) f_p f_{p'}\, W_{pp'|kk'}. \tag{81}$$

In the LHS we can use the expression for the derivative of a product:

$$\text{LHS} = \left[\int \partial_\mu(\psi_k k^\mu f_k)\frac{d^3\mathbf{k}}{k^0} - \int (\partial_\mu \psi_k) k^\mu f_k \frac{d^3\mathbf{k}}{k^0}\right] \tag{82}$$

We can bring the ∂_μ derivative on coordinate outside the integral in the first addend, which leads to the transport equation:

$$\begin{aligned}
&\partial_\mu \int \psi_k k^\mu f_k \frac{d^3\mathbf{k}}{k^0} - \int (\partial_\mu \psi_k) k^\mu f_k \frac{d^3\mathbf{k}}{k^0} = \\
&= \frac{1}{4}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}(\psi_k + \psi_{k'} - \psi_p - \psi_{p'}) f_p f_{p'}\, W_{pp'|kk'}
\end{aligned} \tag{83}$$

If ψ_k depends only on momentum $\psi_k = \psi_k(p_k)$, $\partial_\mu \psi_k = 0$ and one gets

$$\psi_k \partial_\mu \left[\int k^\mu f_k \frac{d^3\mathbf{k}}{k^0}\right] = \frac{1}{4}\int \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}\frac{d^3\mathbf{p}}{p^0}\frac{d^3\mathbf{p}'}{p'^0}(\psi_k + \psi_{k'} - \psi_p - \psi_{p'}) f_p f_{p'}\, W_{pp'|kk'} \tag{84}$$

In particular, choosing suitable functions ψ_k which are summation invariant, we obtain macroscopic conservation laws. For instance considering $\psi_k = 1$ and $\psi_k = k^\nu$:

$$\psi_k = 1 \quad \Rightarrow \quad \partial_\mu N^\mu = 0 \tag{85}$$

$$\psi_k = k^\nu \quad \Rightarrow \quad \partial_\mu T^{\mu\nu} = 0 \tag{86}$$

This result, as shown in section 3, encodes the conservation laws for the first and second moments of the phase-space distribution function. Higher moments are not conserved because they correspond to non-linear terms in ψ_k that give $\psi_k + \psi_{k'} - \psi_p - \psi_{p'} \neq 0$.

3 Macroscopic description

In order to make a connection to the macroscopic description we introduce the moments of the one-particle distribution function, defined as:

$$T^{\alpha\beta\ldots\gamma\delta} = \int p^\alpha p^\beta \ldots p^\gamma p^\delta f(x,p) \frac{d^3\mathbf{p}}{p^0} \tag{87}$$

- First moment: particle current

$$N^\mu(x) = \int \frac{d^3\mathbf{p}}{p^0}\, p^\mu\, f(x,p) \tag{88}$$

- Second moment: energy-momentum tensor

$$T^{\mu\nu}(x) = \int \frac{d^3\mathbf{p}}{p^0}\, p^\mu p^\nu\, f(x,p) \tag{89}$$

- Higher moments can also be defined, for example:

$$T^{\mu\nu\rho}(x) = \int \frac{d^3\mathbf{p}}{p^0}\, p^\mu p^\nu p^\rho\, f(x,p) \tag{90}$$

  which do not have a straightforward physical interpretation.
  For instance if one contracts two indices:

$$T^{\mu\nu}_{\nu}(x) = T^{\mu\nu\rho} g_{\nu\rho} = \int \frac{d^3\mathbf{p}}{p^0}\, p^\mu p^\nu p_\nu\, f(x,p) = m^2 N^\mu \tag{91}$$

We are interested in the case where external forces vanish $K^\mu = 0$, where we can derive conservation or balance equations for the macroscopic quantities N^μ, $T^{\mu\nu}, T^{\mu\nu\rho}$, choosing respectively $\psi = 1$, $\psi = p^\mu$, $T^{\mu\nu} = p^\mu p^\nu$.

3.1 Particle number conservation for a single-component gas

Choosing $b^\mu(x) = 0$ and $a_k(x) = a(x)$, we get simply $\psi = a(x)$. This choice of ψ obviously fulfils the requirement of the Lemma proved above, therefore

$$F[x] = \int \frac{d^3\mathbf{k}}{k^0} a(x) C(x,k) = 0 \tag{92}$$

Hence, due to the arbitrariness of $a(x)$:

$$\int \frac{d^3\mathbf{p}_k}{p^0_k} C_{kl}(x,p_k) = 0 \tag{93}$$

and using the transport equation $k^\mu \partial_\mu f(x,k) = C(x,k)$ we get:

$$0 = \int \frac{d^3\mathbf{k}}{k^0} k^\mu \partial_\mu f(x,k) = \partial_\mu \left[\int \frac{d^3\mathbf{k}}{k^0} k^\mu f(x,k) \right] \tag{94}$$

By definition, the particle 4-current is:

$$N^\mu = \int \frac{d^3\mathbf{k}}{k^0} k^\mu f(x,k) \tag{95}$$

Hence, Eq. (94) is the conservation equation for the 4-current:

$$\partial_\mu N^\mu(x) = 0. \tag{96}$$

3.2 Energy-momentum conservation for a single-component gas

We use exactly the same strategy used to prove the particle-number conservation, simply choosing $a = 0$, $b^\mu = \delta^{\mu\nu}$:

$$\psi(x,k) = k^\nu \tag{97}$$

and the conservation equation becomes:

$$F[x] = \int \frac{d^3\mathbf{k}}{k^0} k^\nu C(x,k) = 0 \tag{98}$$

And using again the transport equation one gets the macroscopic conservation law for the energy-momentum tensor:

$$\int \frac{d^3\mathbf{k}}{k^0} k^\nu k^\mu \partial_\mu f(x,k) = 0 \tag{99}$$

By using the definition of the energy momentum tensor, one immediately finds:

$$\partial_\mu T^{\mu\nu}(x) = 0, \tag{100}$$

In particular, if $\nu = 0$ one gets the conservation of T^{00}, i.e. of the energy density, while for T^{0i} the conservation of momentum with $i = 1, 2, 3$.

3.3 Evolution equation for the third-order moment

By choosing $\psi = k^\mu\, k^\nu$ one can see that it is not possible to get zero from the RHS of Eq. (84) and therefore to get a conservation law for the third order moment. Notice that in this case ψ is not the the form required by the Lemma, therefore we do not have any trivial conservation.

$$k^{\mu\nu} = \frac{1}{2}\int (p^\mu p^\nu + p'^\mu p'^\nu - k^\mu k^\nu - k'^\mu k'^\nu) f_k f_{k'} \mathcal{F}\sigma d\Omega \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0}. \tag{101}$$

It is possible to prove that:

$$\partial_\gamma T^{\alpha\beta\gamma} = k^{\alpha\beta} \tag{102}$$

Notice that $k^\alpha_\alpha = 0 \implies \partial_\beta N^\beta = 0$.

3.4 Entropy 4-flow

If we choose $\psi = 1 - \log(f)$, one gets the equation:

$$\partial_\mu S^\mu = \zeta \tag{103}$$

where

$$S^\mu = -\int \frac{d^3\mathbf{k}}{k^0} k^\mu f_k \left[\log(f_k) - 1\right]. \tag{104}$$

The entropy production rate ζ reads as (Exercise 1):

$$\zeta = -\frac{1}{4}\int dK\, dK'\, dP\, dP' \left[\log\left(\frac{f_k f_{k'}}{f_p f_{p'}}\right) - \frac{f_k f_{k'}}{f_p f_{p'}} + 1\right] f_p f_{p'}\, W_{pp'|kk'} \tag{105}$$

Notice that the f are non-negative, and the sign of the integrand depends solely on the function $1 + \log x - x$, with $x = f_k f_{k'}/(f_p f_{p'})$. It is easy to study this function and to find:

$$\begin{cases} 1 + \log(x) - x < 0 & \text{for } x > 0, x \neq 1 \\ 1 + \log(x) - x = 0 & \text{for } x = 1 \end{cases} \tag{106}$$

This means that $\zeta \geq 0$ always, and in particular $\zeta = 0$ if and only if $f_p f_{p'} = f_k f_{k'}$.
It can also be proved that:

$$\zeta = \frac{1}{4}\int f_k\, f_{k'}\, \log\left(\frac{f_p f_{p'}}{f_k f_{k'}}\right)\left(\frac{f_p f_{p'}}{f_k f_{k'}} - 1\right)\mathcal{F}\sigma d\Omega \frac{d^3\mathbf{k}}{k^0}\frac{d^3\mathbf{k}'}{k'^0} \tag{107}$$

The balance equation takes the shape $\partial_\mu S^\mu = \zeta \geq 0$, which is the equation for the entropy 4-flow, with ζ being the entropy production rate. This is the relativistic formulation of the of the H theorem, and it represents the 2nd law of thermodynamics.

Using the expression for S^μ in Eq.(104) and evaluating $\partial_\mu S^\mu$ we obtain

$$\partial_\mu S^\mu = -\int \frac{d^3\mathbf{k}}{k^0} k^\mu \partial_\mu \{f_k \left[\log(f_k) - 1\right]\} = -\int \frac{d^3\mathbf{k}}{k^0} k^\mu \left(\partial_\mu f_k\right) \log(f_k) \tag{108}$$

Where we have used the fact that $\partial_\mu \{f_k \left[\log(f_k) - 1\right]\} = (\partial_\mu f_k)\log(f_k)$. Finally, using the Boltzmann equation $k^\mu \partial_\mu f_k = C(x, k)$ allows us to replace the streaming term by the collision integral. Hence:

$$\partial_\mu S^\mu = -\int \frac{d^3\mathbf{k}}{k^0} k^\mu C[f_k] \log(f_k). \tag{109}$$

Therefore, the entropy is generated by the microscopic interactions encoded in $C[f_k]$: once $C[f_k] = 0$, $\partial_\mu S^\mu = 0$, even though it not true the opposite (vanishing entropy rate does not mean vanishing collision integral).

Equilibrium state

If we assume that the distribution function tends to a definite limit as time progresses, the state of the system develops into an equilibrium state. In this steady state the entropy of the system attains its maximum value.

A necessary condition for equilibrium is therefore that the entropy production vanishes everywhere in space–time.

Local and global equilibrium

By studying the entropy production rate, we have found that $\zeta \geq 0$ always and in particular

$$\partial_\mu S^\mu = 0 \iff f_k f_{k'} = f_p f_{p'}. \tag{110}$$

By taking the logarithm of both sides one gets:

$$\log f_k + \log f_{k'} = \log f_p + \log f_{p'} \tag{111}$$

This means that, for the equilibrium distribution function, $\log f_k^{eq}$ is a summational invariant quantity and, as we had previously said, this means that it must be of the form:

$$\log f_k = a(x) + b_\mu(x) k^\mu \tag{112}$$

which is trivially solved to:

$$f_k \propto \exp[a(x) + b_\mu(x) k^\mu]. \tag{113}$$

Notice that this condition allows the entropy production to vanish, as well as the collision integral. It can be shown that it corresponds to the case of local thermal equilibrium. However, if one wants it to solve the Boltzmann Equation, the LHS of the BE should vanish as well:

$$k^\mu \partial_\mu \exp[a(x) + b_\nu(x) k^\nu] + m F^\mu \frac{\partial}{\partial k^\mu} \exp[a(x) + b_\nu(x) k^\nu] = 0 \tag{114}$$

If there are no external forces $F^\mu = 0$ and the following condition must hold:

$$k^\mu \partial_\mu a(x) + k^\nu k^\mu \partial_\mu b_\nu = 0 \tag{115}$$

Due to the arbitrariness of k^μ:

$$\partial_\mu a(x) = 0 \implies a(x) = a = \text{const.} \tag{116}$$

$$\partial_\mu b_\nu + \partial_\nu b_\mu = 0 \tag{117}$$

The latter is the Killing equation, whose most general solution is:
$b^\mu(x) = b^\mu + \omega^{\mu\nu} x_\nu, \qquad \text{with}\ \omega^{\mu\nu} = -\omega^{\nu\mu}$.

In particular, if the system is not rotating: $\omega^{\mu\nu} = 0$ and the global equilibrium distribution becomes:

$$f(x, k) \propto \exp[a + b_\mu k^\mu], \tag{118}$$

with constant a and b_μ. The derivation of a and b_μ is carried out in Exercise 2.

References

[1] C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications*, Birkhäuser, 2002.

[2] S. R. de Groot, W. A. van Leeuwen, and C. G. van Weert, *Relativistic Kinetic Theory: Principles and Applications*, North-Holland, 1980.

[3] G. S. Denicol and D. H. Rischke, *Microscopic Foundations of Relativistic Fluid Dynamics*, Springer, 2021.

[4] W. Florkowski, *Phenomenology of Ultra-Relativistic Heavy-Ion Collisions*, World Scientific, 2010.