Lecture notes BIP Timisoara 2026

CONTENTS

INTRODUCTION

Relativistic hydrodynamics from Statistical Quantum Mechanics (SQM)

$\hat{\rho}$ = density operator $\quad \hat{\rho}^{\dagger} = \hat{\rho} \quad \mathrm{Tr}\,\hat{\rho} = 1 \quad \hat{\rho}^2 \le \hat{\rho}$

$T^{\mu\nu} \to \mathrm{Tr}(\hat{\rho}\,\hat{T}^{\mu\nu}(x)) \qquad \partial_\mu T^{\mu\nu} \longrightarrow \mathrm{Tr}(\hat{\rho}\,\partial_\mu \hat{T}^{\mu\nu}) = 0$

How to obtain $\hat{\rho}$?

GLOBAL EQUILIBRIUM

Entropy in SQM $\qquad S = -\mathrm{Tr}\,(\hat{\rho} \log \hat{\rho})$

$\hat{\rho}$ can only depend on globally constant

$\rightsquigarrow$ max entropy (basic principle of S(Q)M)

10 constants $\quad P_0^{\mu} \quad J_0^{\mu\nu}$ mean values + some charges Q_{0i}

Each constant $\longleftrightarrow$ Lagrange multiplier

$$F[\hat{\rho}] = -\mathrm{Tr}(\hat{\rho}\log\hat{\rho}) - b_\mu\left(\mathrm{Tr}(\hat{\rho}\hat{P}^\mu) - P_0^\mu\right) + \frac{\varpi_{\mu\nu}}{2}\left(\mathrm{Tr}(\hat{\rho}\hat{J}^{\mu\nu}) - J_0^{\mu\nu}\right)$$
$$+ \sum_i \zeta_i\left(\mathrm{Tr}(\hat{\rho}\hat{Q}_i) - Q_{0i}\right) + \lambda\left(\mathrm{Tr}(\hat{\rho}) - 1\right)$$

$$\frac{\delta F}{\delta \hat{\rho}} = 0$$

Solution $$\hat{\rho} = \frac{e^{-b\cdot\hat{P} + \frac{1}{2}\varpi:\hat{J} + \sum_i \zeta_i \hat{Q}_i}}{Z}$$

$$Z = \mathrm{Tr}\left(e^{-b\cdot\hat{P} + \frac{1}{2}\varpi:\hat{J} + \sum_i \zeta_i \hat{Q}_i}\right)$$

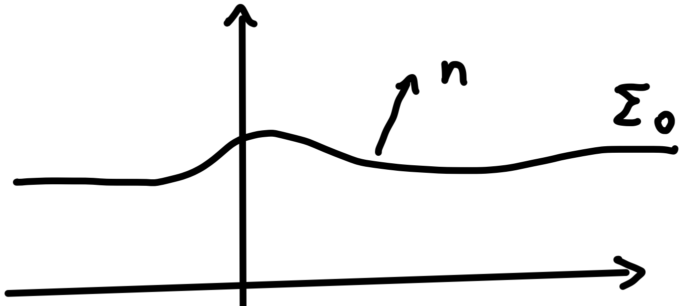
b = constant future-timelike vector

ϖ = Constant anti-symmetric tensor

Problem : prove it (T)

LOCAL EQUILIBRIUM

Maximize entropy with DENSITIES of conserved currents



$$n_\mu \mathrm{Tr}(\hat\rho\, \hat T^{\mu\nu}(x)) = n_\mu T^{\mu\nu}_{\text{given}}(x) \quad \text{on } \Sigma_0$$

$$n_\mu \mathrm{Tr}(\hat\rho\, \hat j^{\mu}(x)) = n_\mu j^{\mu}_{\text{given}}(x) \quad \text{on } \Sigma_0$$

$$F[\hat\rho] = -\mathrm{Tr}(\hat\rho \log\hat\rho) - \int_{\Sigma_0} d\Sigma_\mu \Big[\beta_\nu(x)\big(\mathrm{Tr}(\hat\rho\, \hat T^{\mu\nu}(x)) - T^{\mu\nu}_{\text{given}}\big) -$$

$$\zeta(x)\big(\mathrm{Tr}(\hat\rho\, \hat j^{\mu}(x)) - j^{\mu}_{\text{given}}\big)\Big] + \lambda(\mathrm{Tr}(\hat\rho) - 1)$$

β, ζ solutions of the constraints

Solution:

$$\hat\rho_{LE}(\Sigma_0) = \exp\Big[-\int_{\Sigma_0} d\Sigma_\mu\, \hat T^{\mu\nu}\beta_\nu - \zeta\, \hat j^\mu\Big] \Big/ Z$$

Problem: prove that β_μ is a Killing vector at G.E. in G.R. (T)

Physical meaning of β and ζ?

$$S = -\mathrm{Tr}(\hat{\rho}_{LE} \log \hat{\rho}_{LE}) = \log Z + \int d\Sigma_\mu \beta_\nu \langle \hat{T}^{\mu\nu} \rangle_{LE} - \int d\Sigma_\mu \zeta \langle \hat{j}^\mu \rangle_{LE}$$

Suppose β is constant and so ζ

$$\Rightarrow S = \log Z + \beta_\nu \int d\Sigma_\mu \langle \hat{T}^{\mu\nu} \rangle_{LE} - \zeta \int d\Sigma_\mu \langle \hat{j}^\mu \rangle_{LE}$$

$$= \log Z + \beta_\nu \langle \hat{P}^\nu \rangle_{LE} - \zeta \langle \hat{Q} \rangle_{LE}$$

Comparing with classical expression $TS = \log Z + u \cdot P - \mu Q$

$$\beta^\mu = \frac{1}{T} u^\mu \qquad \zeta = \frac{\mu}{T} \qquad \forall x \,!$$

Problem : prove that $\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu$ at global equilibrium (T)

Problem : different cases for b, ϖ ; Tolman's law (T)

Calculate $T^{\mu\nu}_{LE}(x) = \mathrm{Tr}\left(\hat{\rho}_{LE}\, \hat{T}^{\mu\nu}(x)\right)$

$$\mathrm{Tr}\left(\hat{\rho}_{LE}\, \hat{T}^{\mu\nu}(x)\right) = \frac{1}{Z_{LE}} \mathrm{Tr}\left(e^{-\int_{\Sigma_0} d\Sigma_\rho \left(\hat{T}^{\rho\sigma}\beta_\sigma - \zeta \hat{j}^\rho\right)} \hat{T}^{\mu\nu}(x)\right)$$

Idea: mean value in x mostly determined by β, ζ around x

$$e^{-\int_{\Sigma_0} d\Sigma_\rho(y) \hat{T}^{\rho\sigma}(y)\beta_\sigma(y)} \cong e^{-\int_{\Sigma_0} d\Sigma_\rho(y)\hat{T}^{\rho\sigma}(y)\left[\beta_\sigma(x) + \partial_\lambda \beta_\sigma(x)(y-x)^\lambda + \ldots\right]}$$

$$= e^{-\beta_\sigma(x) \underbrace{\int_{\Sigma_0} d\Sigma_\rho(y)\hat{T}^{\rho\sigma}(y)}_{\hat{P}^\sigma} + \partial_\lambda \beta_\sigma(x) \int_{\Sigma_0} d\Sigma_\rho(y)(y-x)^\lambda \hat{T}^{\rho\sigma} + \ldots}$$

$$= e^{-\beta_\sigma(x)\hat{P}^\sigma + \text{corrections}}$$

At the lowest order $\quad T^{\mu\nu}_{LE}(x) \cong \mathrm{Tr}\left(\frac{e^{-\beta(x)\cdot\hat{P}}}{Z_0}\hat{T}^{\mu\nu}(x)\right)$

$$\rightarrow \quad T^{\mu\nu}_{LE}(x) \cong A\,\beta^\mu\beta^\nu + B\,g^{\mu\nu}$$

equivalent to $(\rho + p)\,u^\mu u^\nu - p\,g^{\mu\nu}$ Proof $\rightarrow$ tutorial

Constitutive equations at LE

$$T^{\mu\nu}_{LE}(x) = T^{\mu\nu}_{ideal} + \text{quantum corrections}$$

ENTROPY CURRENT

1) $S = -\mathrm{Tr}(\hat\rho \log \hat\rho) \longrightarrow -\mathrm{Tr}(\hat\rho_{LE} \log \hat\rho_{LE})$ $\quad \dfrac{d\hat\rho_{LE}}{dt} \neq 0$

$\dfrac{d\hat\rho}{dt}$

maximized

$\downarrow$ constrained ignorance

2) Extensivity

Let $\quad \hat\rho(\lambda) = \dfrac{1}{Z(\lambda)} \exp\left[-\lambda \int d\Sigma_\mu \hat T^{\mu\nu} \beta_\nu \right]$ $\quad (\zeta = 0$ for simplicity$)$ $\quad \hat\rho \equiv \hat\rho(1)$

$$Z(\lambda) = \mathrm{Tr}\left(\exp\left[-\lambda \int d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu\right]\right) \Rightarrow \frac{\partial \log Z}{\partial\lambda} = \frac{1}{Z(\lambda)}\frac{\partial}{\partial\lambda}\mathrm{Tr}\left(\exp\left[-\lambda\int d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu\right]\right)$$

$$= \frac{\mathrm{Tr}}{Z(\lambda)}\left(\exp\left[-\lambda\int d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu\right](-1)\int d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu\right) = -\int d\Sigma_\mu \langle \hat T^{\mu\nu}\rangle(\lambda)\,\beta_\nu$$

$\langle \hat T^{\mu\nu}\rangle(\lambda)$

mean value with λ

Integrating between 1 and λ_0

$$\int_1^{\lambda_0} d\lambda \frac{\partial}{\partial\lambda} \log Z(\lambda) = \log Z(\lambda_0) - \log Z = -\int_1^{\lambda_0} d\lambda \int d\Sigma_\mu \langle \hat{T}^{\mu\nu}\rangle(\lambda)\, \beta_\nu = \int d\Sigma_\mu \left(\int_{\lambda_0}^1 d\lambda \langle \hat{T}^{\mu\nu}\rangle(\lambda)\, \beta_\nu \right)$$

If λ_0 such that $\log Z(\lambda_0) = 0$

$$\Rightarrow \quad \log Z = \int d\Sigma_\mu \phi^\mu \quad \text{with} \quad \phi^\mu = \int_1^{\lambda_0} d\lambda \langle \hat{T}^{\mu\nu}\rangle(\lambda)\, \beta_\nu$$

Does it exist?

Suppose $\lambda_0 = +\infty \Rightarrow \lim_{\lambda_0 \to \infty} \dfrac{e^{-\lambda_0 \int d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu}}{Z(\lambda)} = |0\rangle\langle 0|$

$|0\rangle = \text{LLS of} \int d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu$

$\boxed{\text{Project: calculate } \phi^\mu \text{ for } \hat{\rho} = e^{-b\cdot\hat{P}}/Z}$

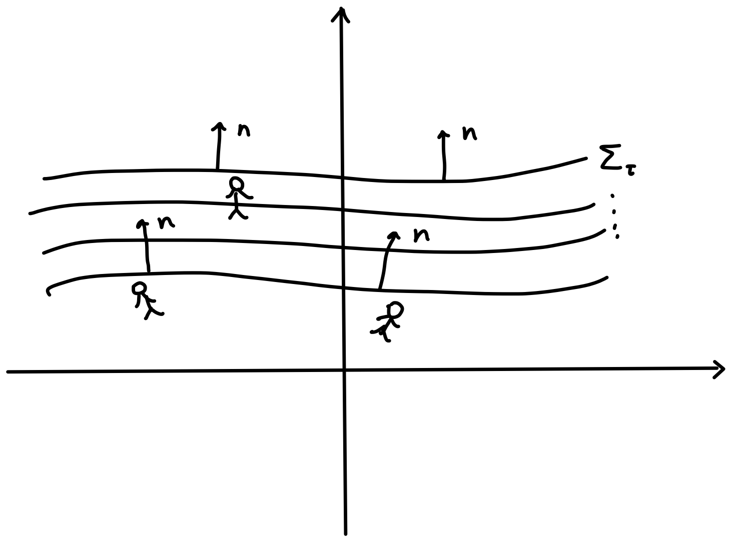
Entropy current

$$S = \int d\Sigma_\mu \, \phi^\mu + T^{\mu\nu}_{LE} \beta_\nu - \zeta j^\mu_{LE}$$

$$n_\mu T^{\mu\nu} = n_\mu T^{\mu\nu}_{LE}$$
$$n_\mu j^\mu = n_\mu j^\mu_{LE}$$

$$S = \int d\Sigma_\mu \, \phi^\mu + T^{\mu\nu} \beta_\nu - \zeta j^\mu$$

$$\Rightarrow \boxed{s^\mu = \phi^\mu + T^{\mu\nu}\beta_\nu - \zeta j^\mu}$$



is it independent of Σ (better: the foliation?)

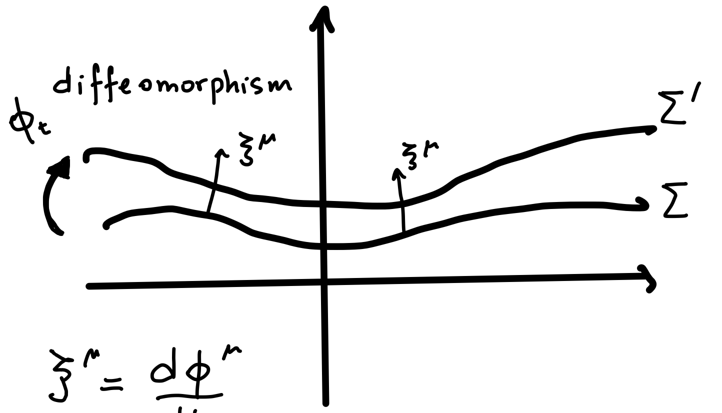
No, because $\zeta(n)$ and $\beta(n)$ both depend on the foliation.

It can be made independent only if there was a privileged foliation

or at global equilibrium.

Entropy production

$$\boxed{\frac{\delta}{\delta\Sigma}\left(\int_\Sigma d\Sigma_\mu V^\mu\right)_\xi = \frac{1}{2}\int d\tilde{S}_{\mu\nu}\,\xi^\nu V^\mu + \int d\Sigma_\mu\, \xi^\mu\, \nabla\cdot V}$$



$$\xi^\mu = \frac{d\phi^\mu}{dt}$$

$$Z_{\phi_t(\Sigma)} - Z_\Sigma = \int_{\phi_t(\Sigma)} d\Sigma_\mu \phi^\mu - \int d\Sigma_\mu \phi^\mu \Longrightarrow \mathcal{L}_\xi \int d\Sigma_\mu \phi^\mu \text{ domain derivative} = \int d\Sigma_\mu\, \xi^\mu\, \nabla\cdot\phi \quad \text{if } \int d\tilde{S} = 0$$

On the other hand:

$$Z_{\phi_t(\Sigma)} = \mathrm{Tr}\left(e^{-\int_{\phi_t(\Sigma)} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu}\right) \cong \mathrm{Tr}\left(e^{-\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \varepsilon\int d\Sigma_\mu \xi^\mu\, \nabla\cdot(\hat{T}^{\mu\nu}\beta_\nu)}\right) \text{ if } \int d\tilde{S} = 0$$

$$\cong \quad Z - \varepsilon\, \mathrm{Tr}\left(e^{-\int\cdots} \int d\Sigma\cdot\xi \;\; \nabla_\mu(\hat{T}^{\mu\nu}\beta_\nu)\right)$$

whence

$$\lim_{\varepsilon\to 0} \frac{Z_\varepsilon - Z}{\varepsilon Z} = \mathcal{L}_\xi(\log Z) = -\int d\Sigma\cdot\xi \;\; \langle\hat{T}^{\mu\nu}\rangle_{LE}\, \nabla_\mu\beta_\nu$$

$$\Rightarrow \quad \mathcal{L}_{\xi} \int d\Sigma_\mu \phi^\mu = \int d\Sigma \cdot \xi \, \nabla \cdot \phi = - \int d\Sigma \cdot \xi \, \langle \hat{T}^{\mu\nu} \rangle_{LE} \nabla_\mu \beta_\nu$$

being ξ arbitrary $\Rightarrow$ $\nabla \cdot \phi = - \langle \hat{T}^{\mu\nu} \rangle_{LE} \nabla_\mu \beta_\nu$

and, with current, $\nabla \cdot \phi = - \langle \hat{T}^{\mu\nu} \rangle_{LE} \nabla_\mu \beta_\nu + \langle \hat{j}^{\mu} \rangle_{LE} \nabla_\mu \zeta$

$$\Rightarrow \nabla \cdot s = \nabla \cdot \left(\phi + T^{\mu\nu} \beta_\nu - j^\mu \right) = \nabla \cdot \phi + \cancel{\nabla_\mu T^{\mu\nu}} \beta_\nu + T^{\mu\nu} \nabla_\mu \beta_\nu - \cancel{\nabla j} \zeta$$

$$- \nabla \zeta \cdot j = - T^{\mu\nu}_{LE} \nabla_\mu \beta_\nu + j^\mu_{LE} \nabla_\mu \zeta + T^{\mu\nu} \nabla_\mu \beta_\nu - \nabla_\mu \zeta \, j^\mu \Rightarrow$$

$$\boxed{\nabla \cdot s = \left(T^{\mu\nu} - T^{\mu\nu}_{LE} \right) \nabla_\mu \beta_\nu - \left(j^\mu - j^\mu_{LE} \right) \nabla_\mu \zeta}$$